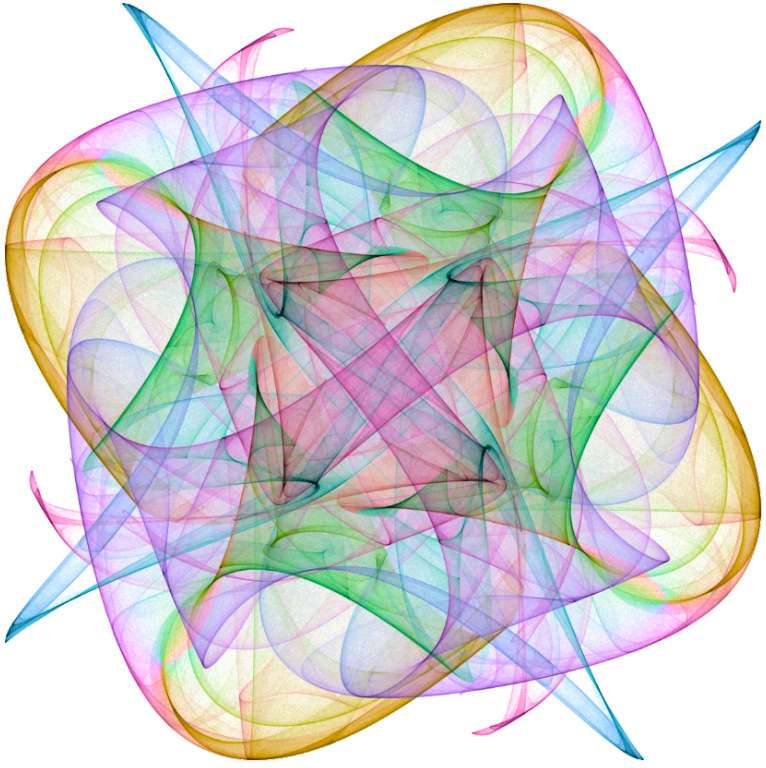
Symmetry and Chaos

- *Vladimir Bulatov*
- *bulatov.org*
- G4G16
- *San Francisco*
- *February 19, 2026*



Experimental fact



When you have a hammer, everything looks like a nail

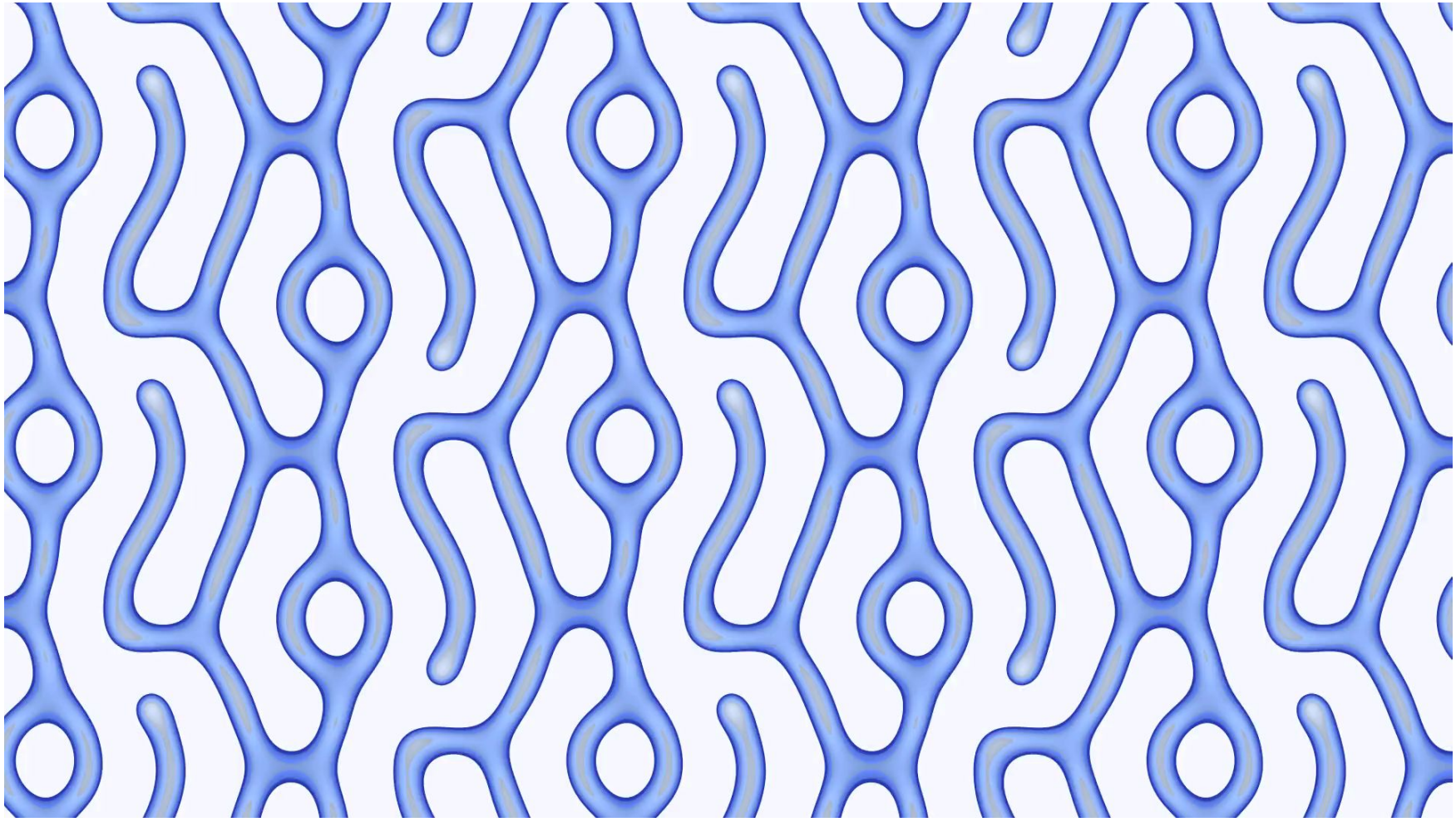
The Hammer - Symmetry



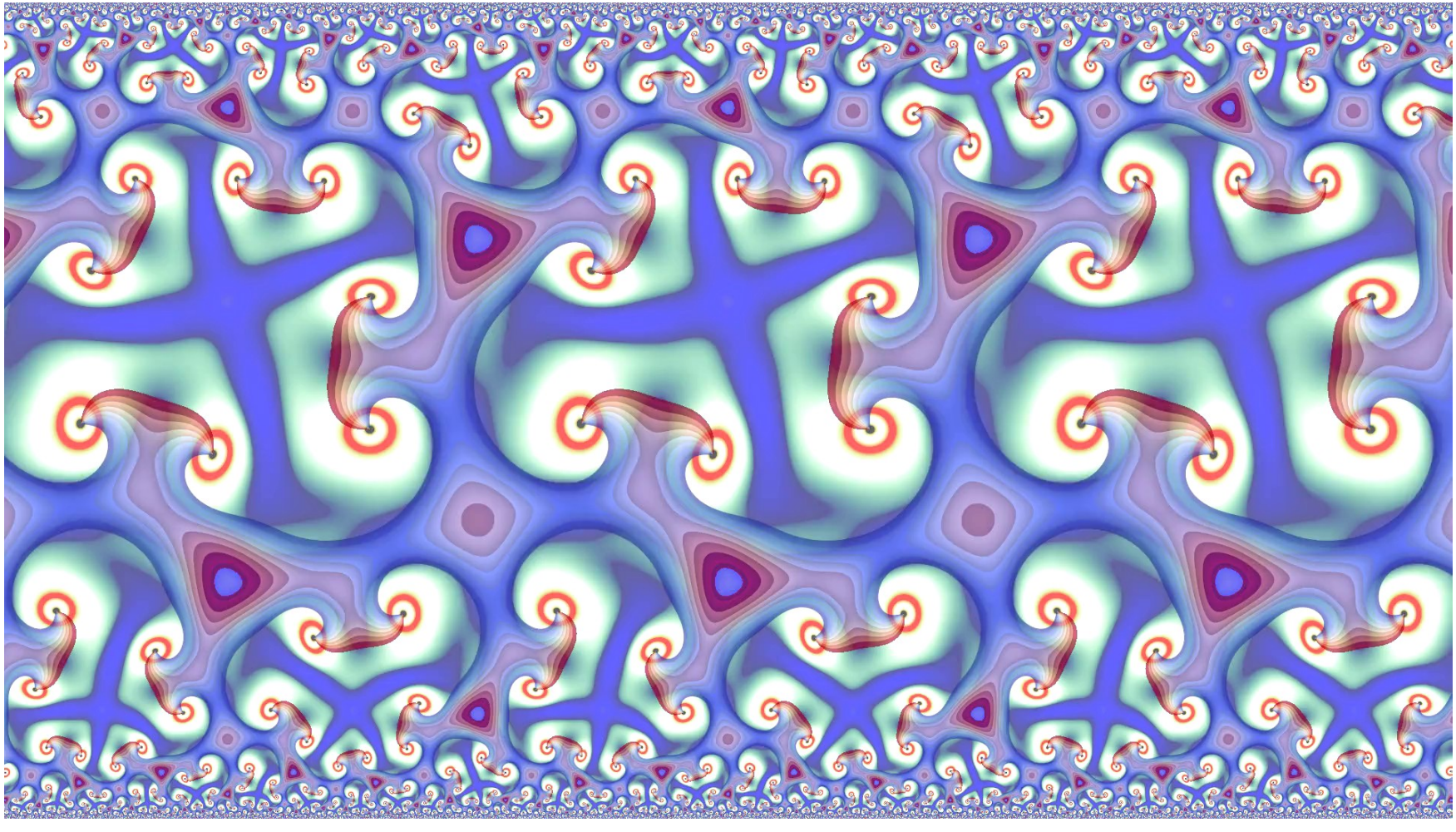
In geometry, **symmetry** is defined as a property of a shape or object that remains invariant (unchanged) under certain transformations.

If you can flip, spin, or slide a shape and it looks exactly as it did before, it is symmetrical.

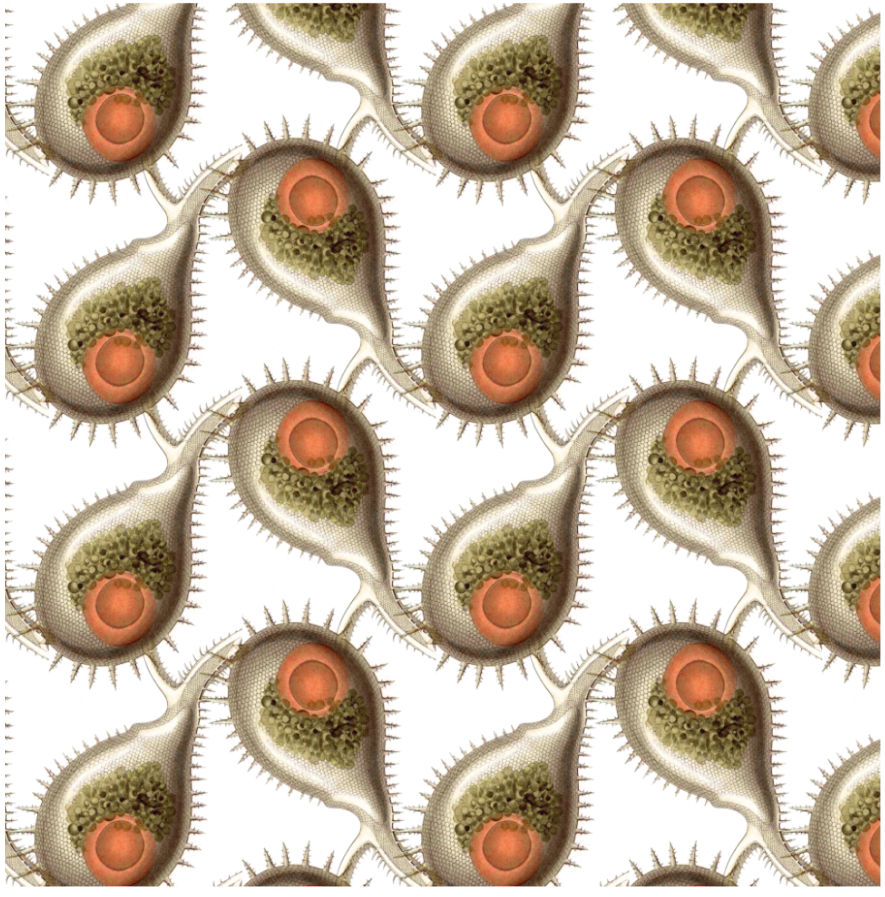
The old victim: SymSim - Symmetry in Gray-Scott Reaction Diffusion

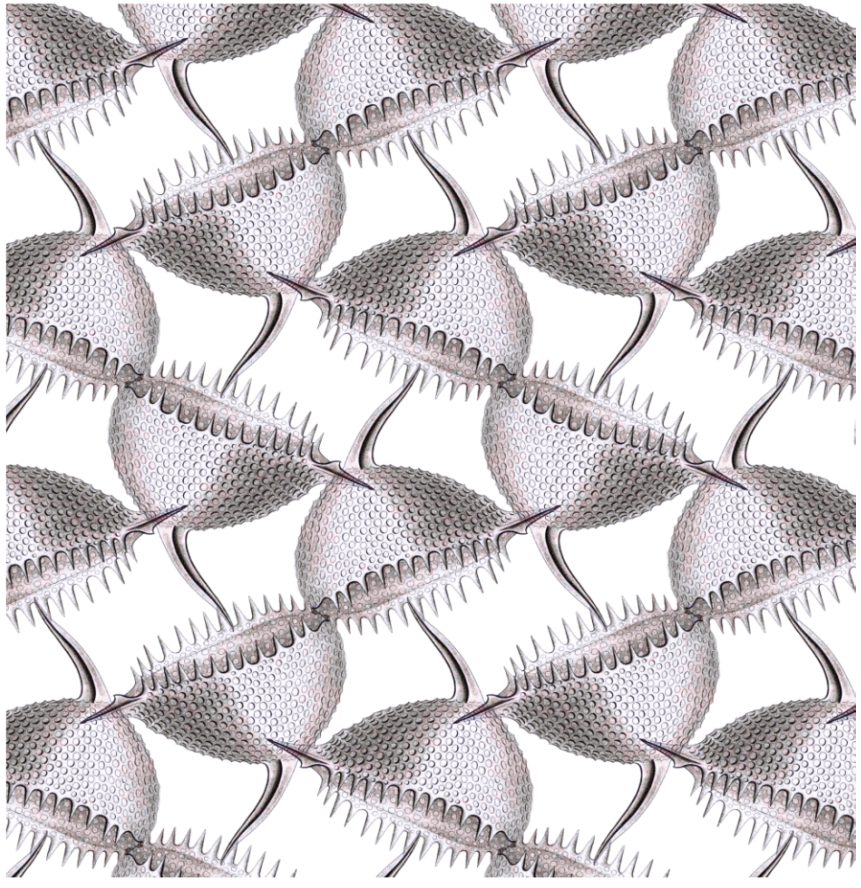


The old victim: SymSim - Symmetry in Ginzburg-Landau equation



The old victim: SymPix - Ernst Haeckel Patterns

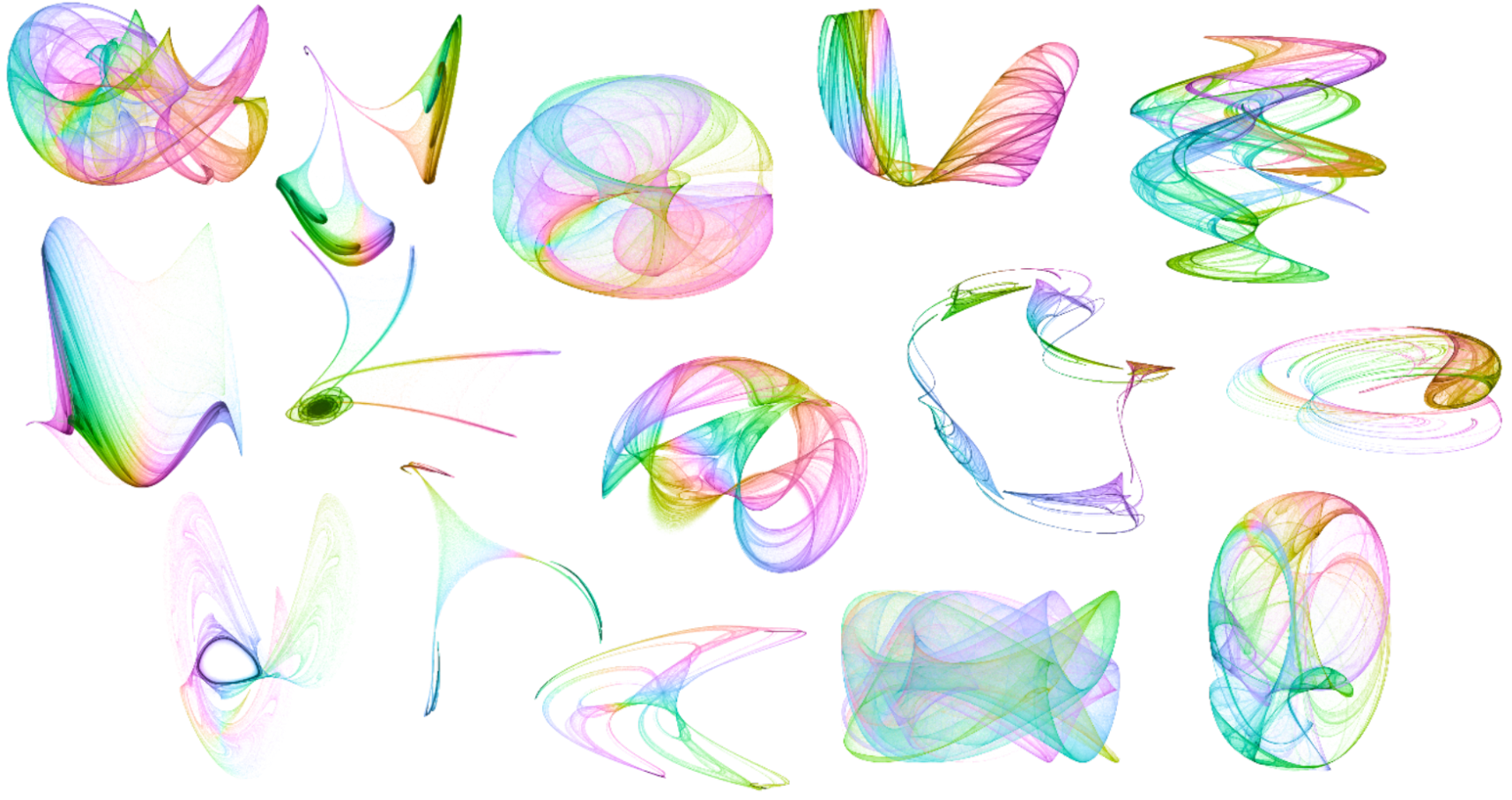




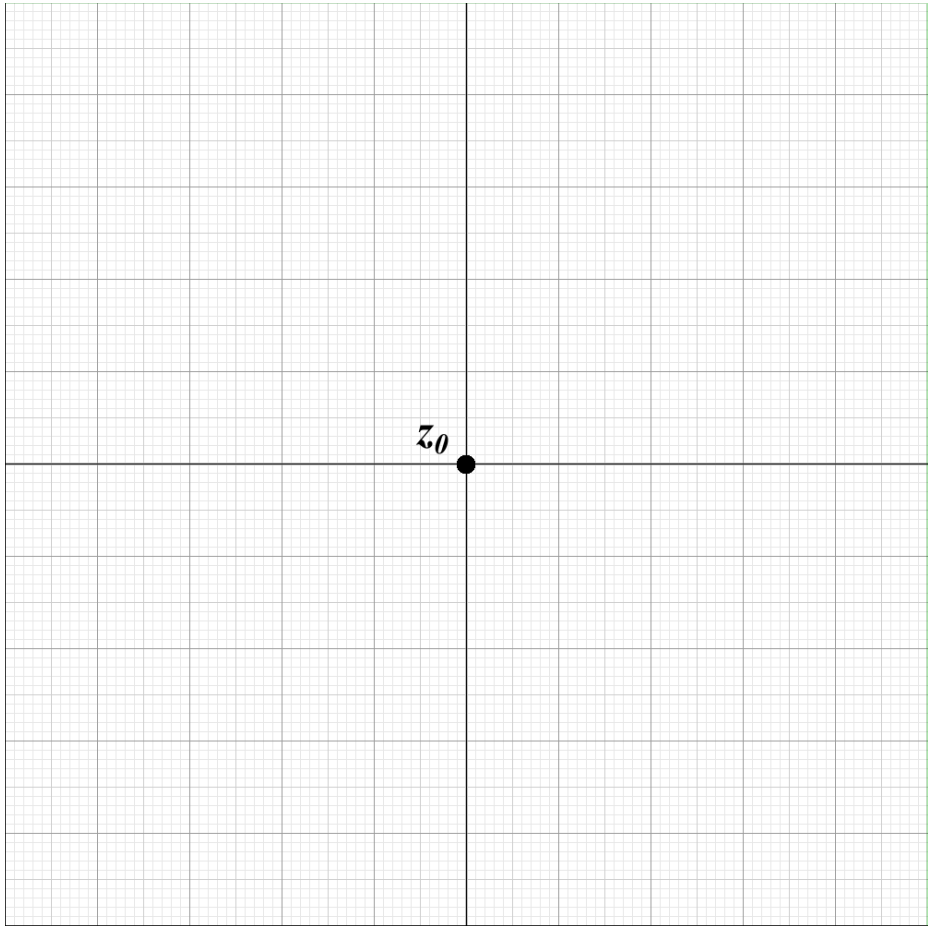
The old victim: M.C.Escher exhibition at Mudec, Milano



The Latest Victims - Chaotic Attractors of Iterated Functions

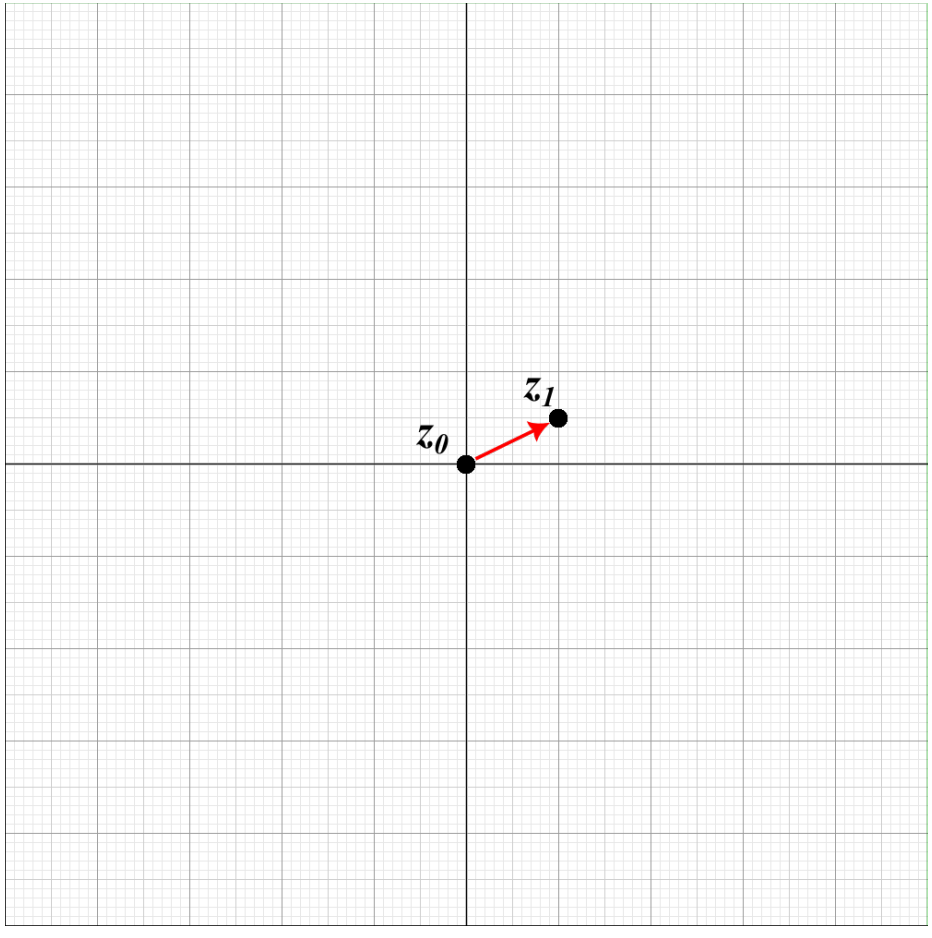


What are Iterations?



Select an arbitrary initial point z_0.

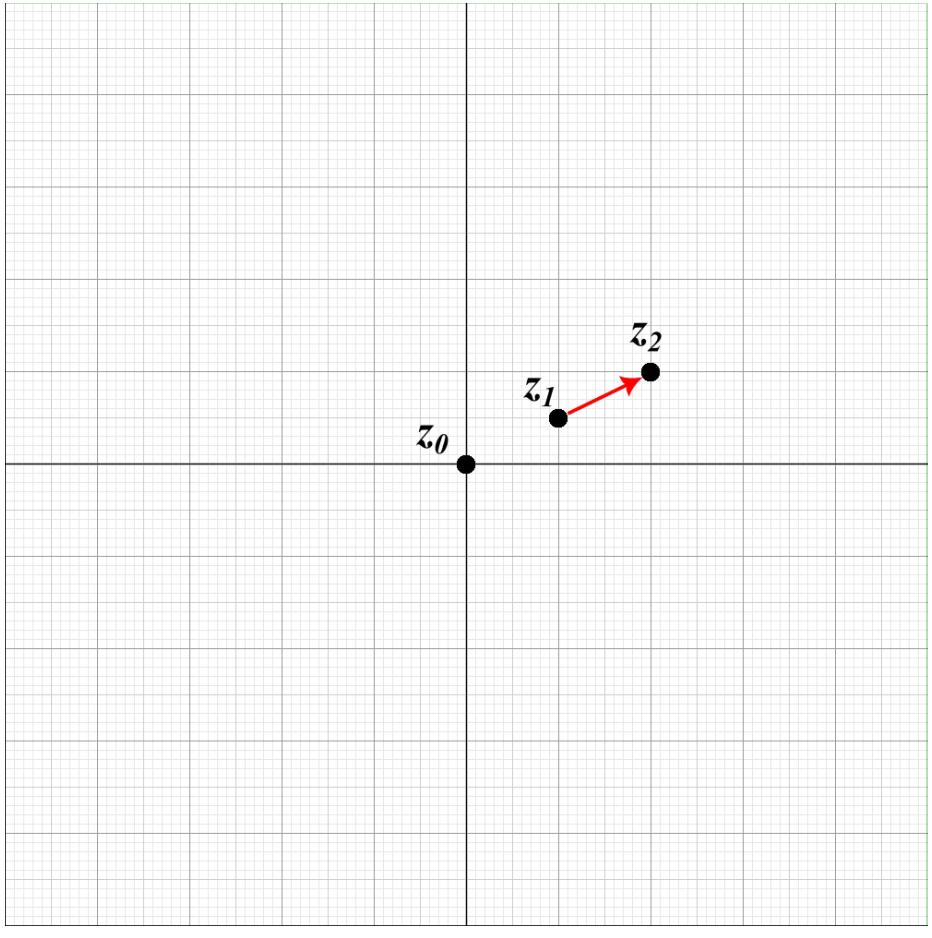
Translation: 1 iterations



Select an arbitrary initial point z_0
Select a function $F(z) = z + p$
Apply the function (iteration):

- $z_1 = F(z_0) = z_0 + p$

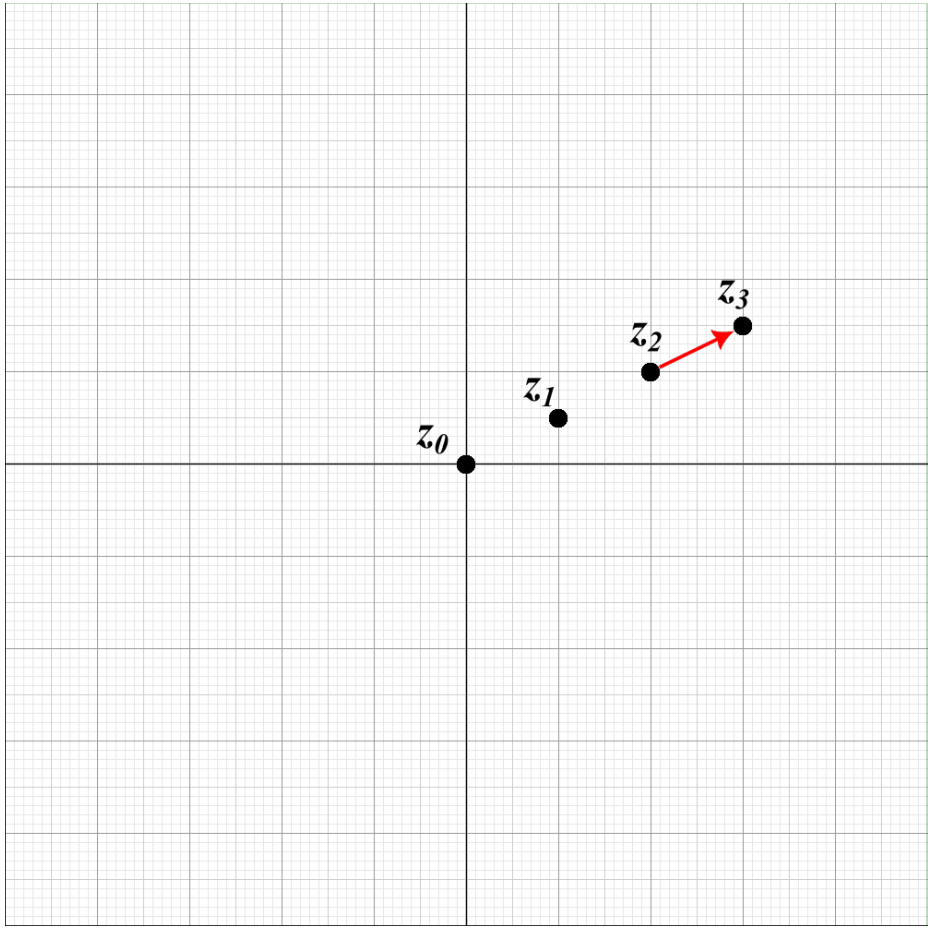
Translation: 2 iterations



Select an arbitrary initial point z_0.
Select a function $F(z) = z + p$
Apply the function (iteration):

- $z_1 = F(z_0) = z_0 + p$
- $z_2 = F(z_1) = z_1 + p$

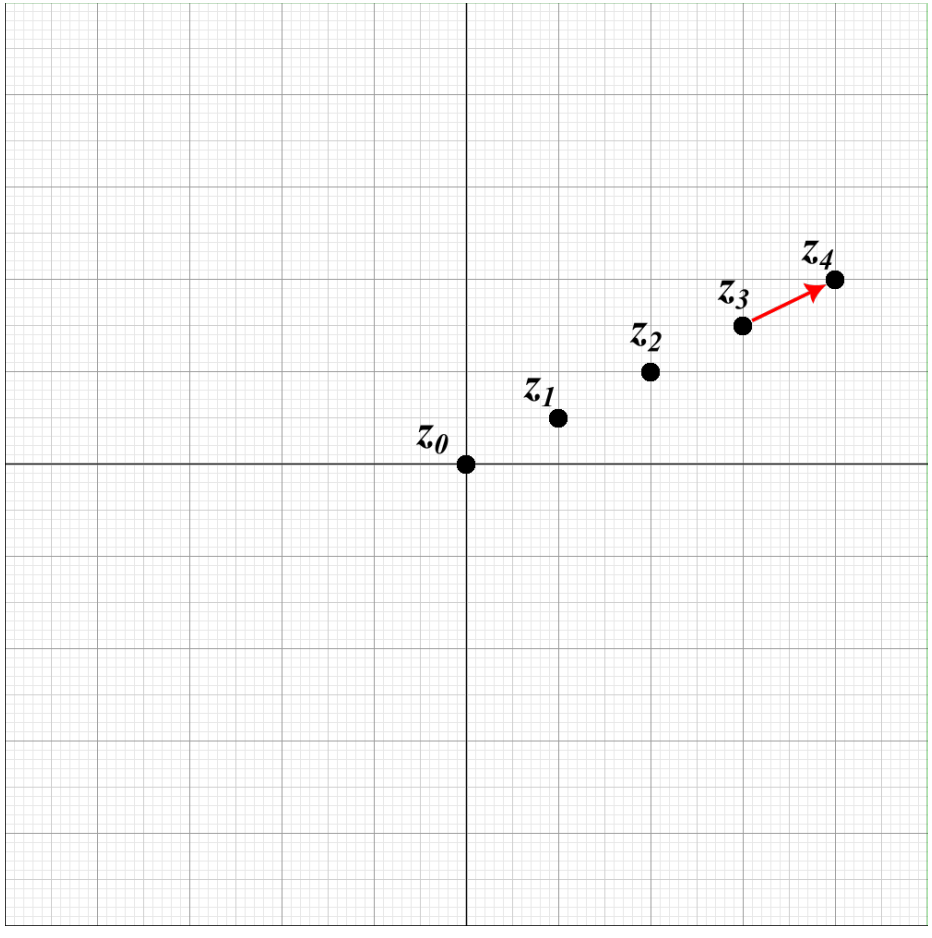
Translation: 3 iterations



Select an arbitrary initial point z_0.
Select a function $F(z) = z + p$
Apply the function (iteration):

- $z_1 = F(z_0) = z_0 + p$
- $z_2 = F(z_1) = z_1 + p$
- $z_3 = F(z_2) = z_2 + p$

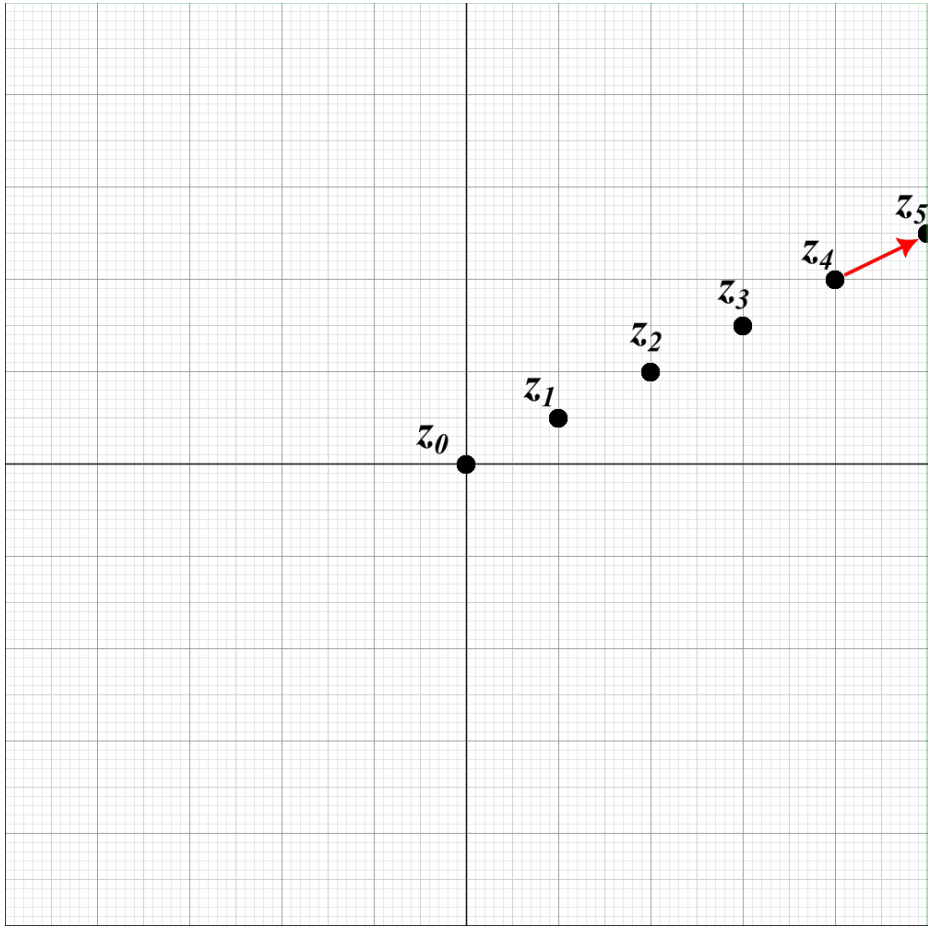
Translation: 4 iterations



Select an arbitrary initial point z_0.
Select a function $F(z) = z + p$
Apply the function (iteration):

- $z_1 = F(z_0) = z_0 + p$
- $z_2 = F(z_1) = z_1 + p$
- $z_3 = F(z_2) = z_2 + p$
- $z_4 = F(z_3) = z_3 + p$

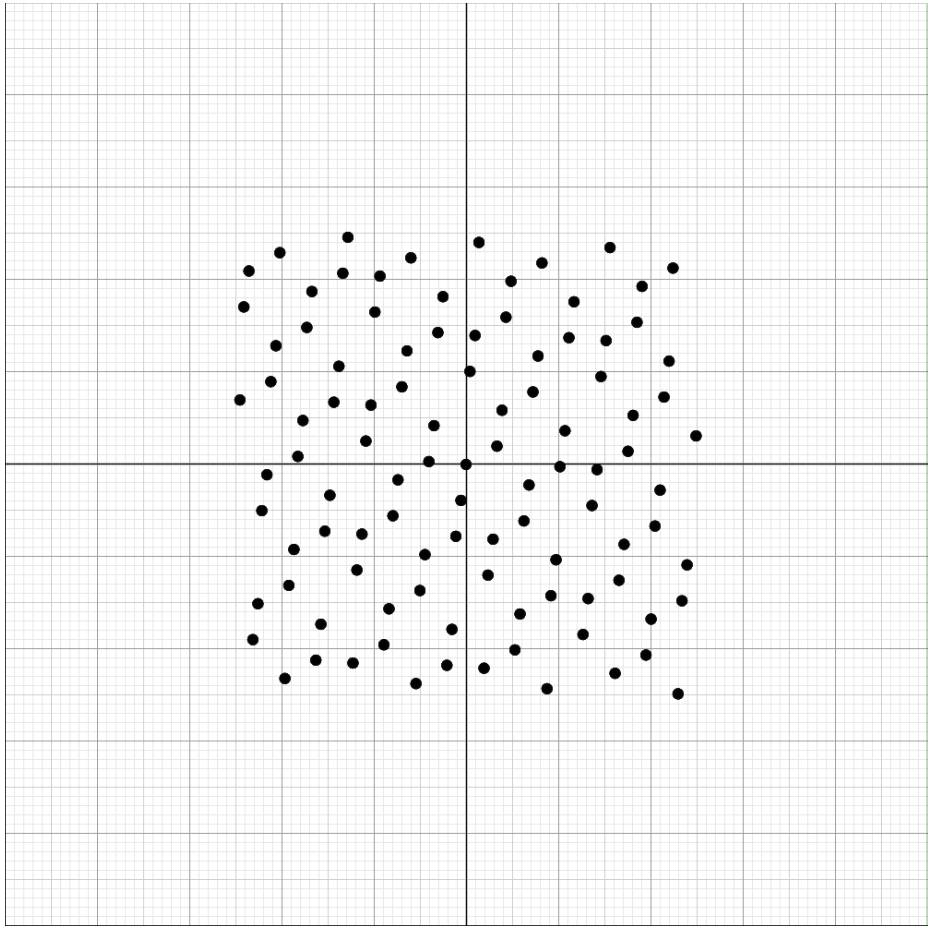
Translation: 5 iterations



Select an arbitrary initial point z_0.
Select a function $F(z) = z + p$
Apply the function (iteration):

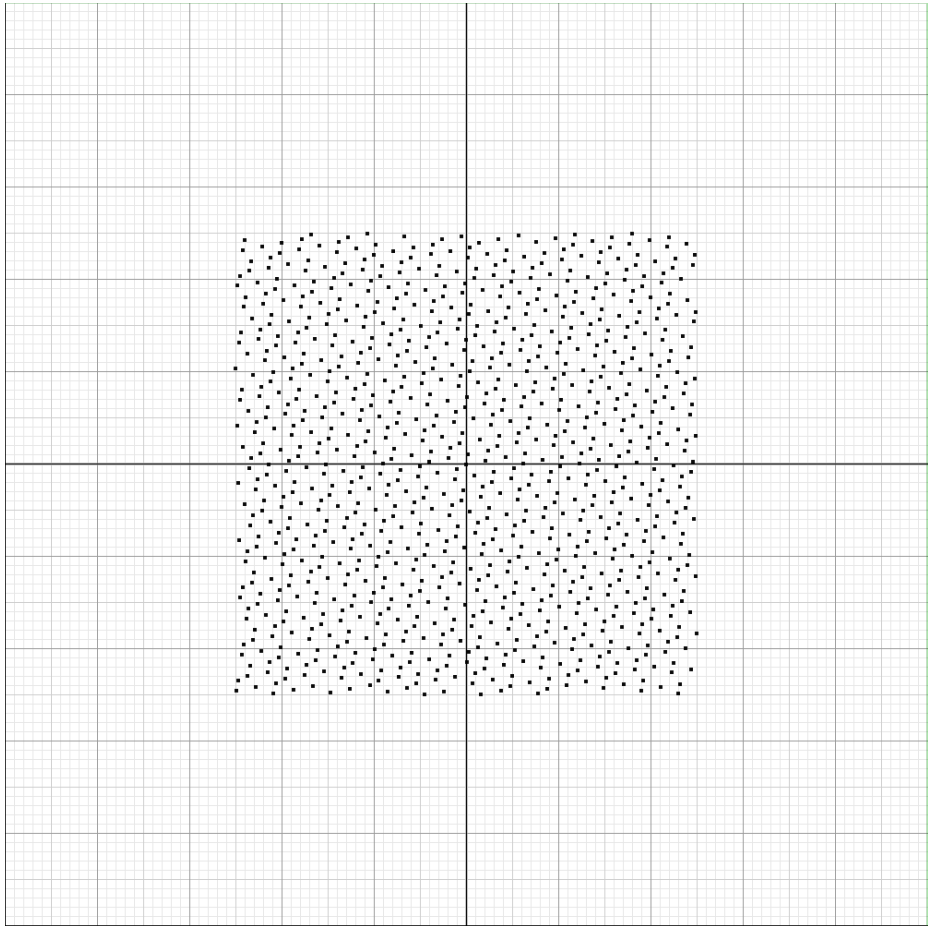
- $z_1 = F(z_0) = z_0 + p$
- $z_2 = F(z_1) = z_1 + p$
- $z_3 = F(z_2) = z_2 + p$
- $z_4 = F(z_3) = z_3 + p$
- $z_5 = F(z_4) = z_4 + p$

100 points



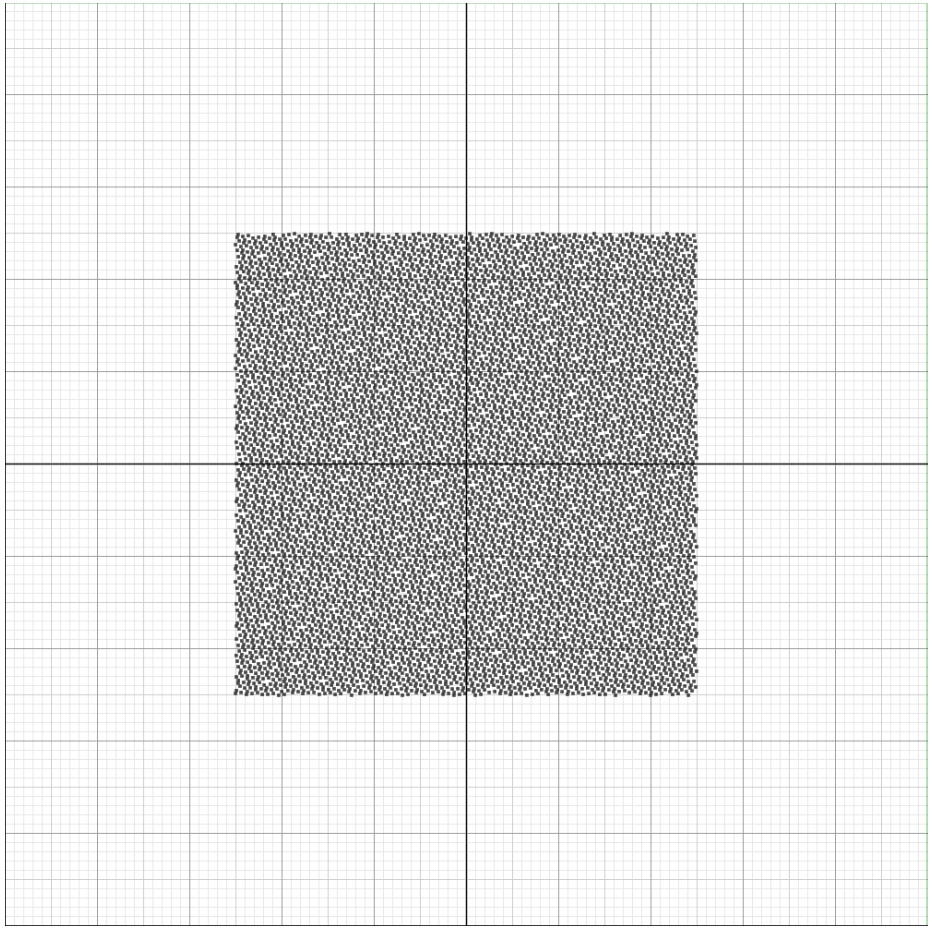
- We can simultaneously calculate many points
- 100 points - too few

1,000 points



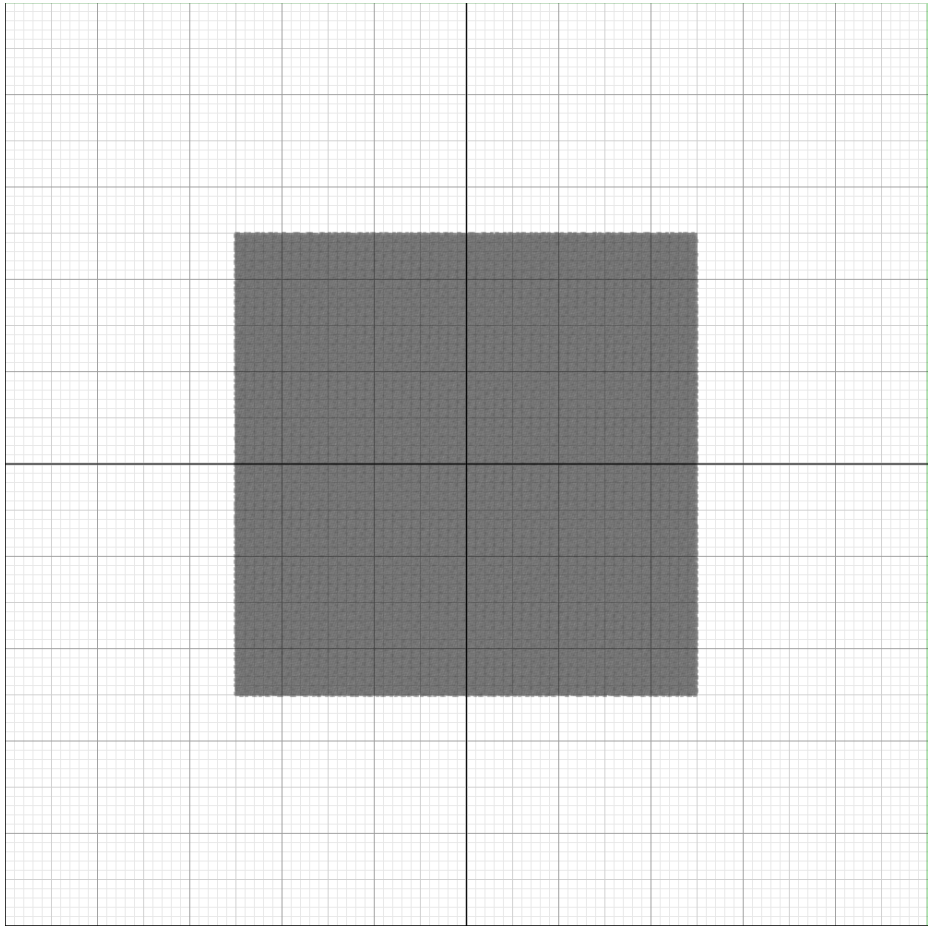
- We can simultaneously calculate many points
- 1000 points - still too few

10,000 points



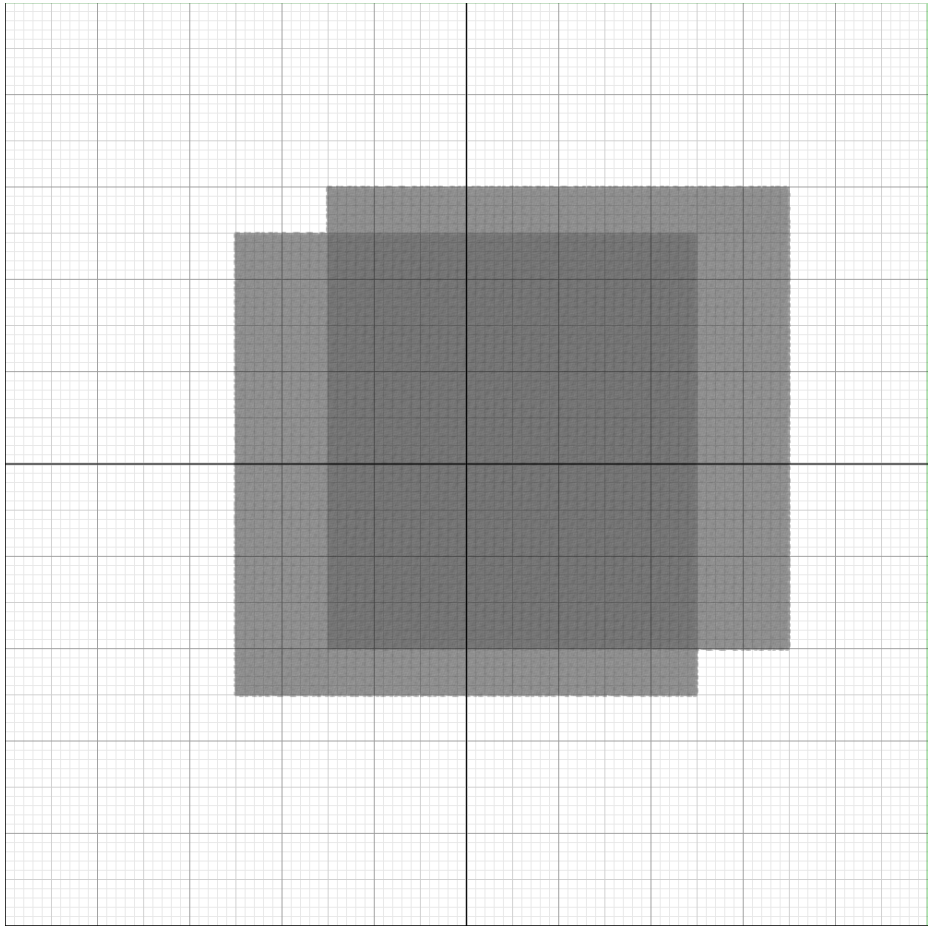
- We can simultaneously calculate many points
- 10,000 points - almost good

100,000 points



- We can simultaneously calculate many points
- 100,000 points - good enough
- 1,000,000 would be even better!

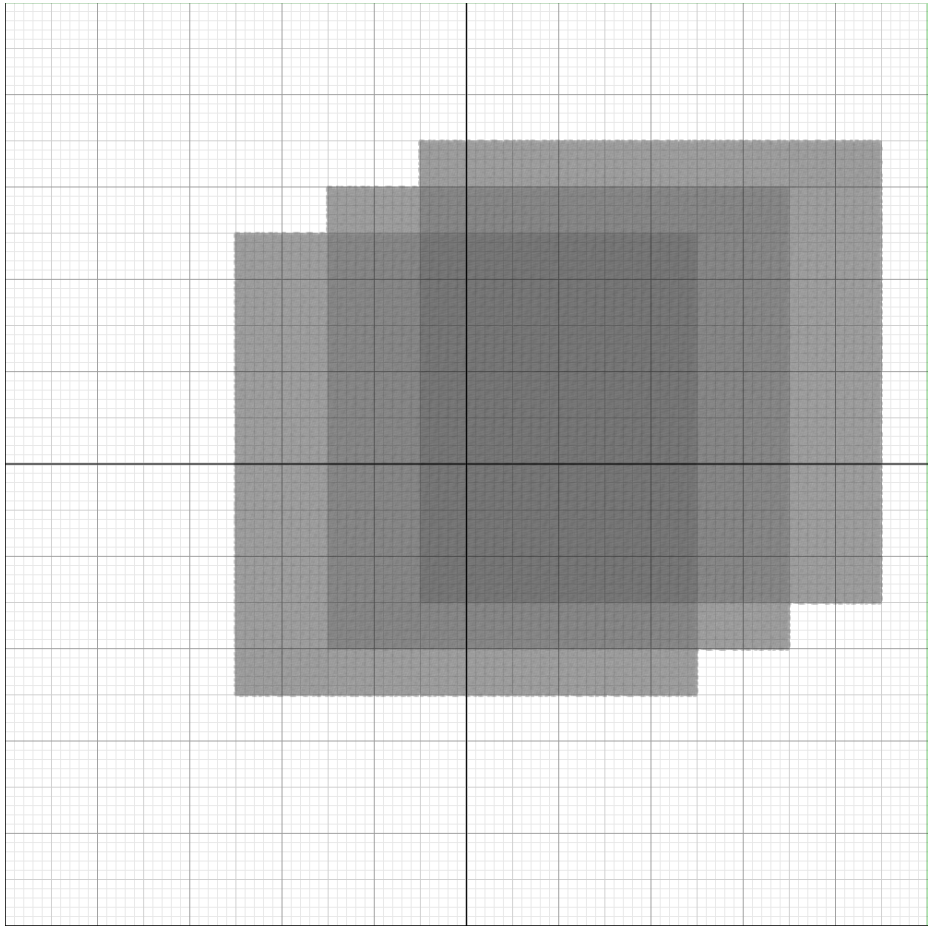
Translation: 100,000 points, 1 iteration



Apply the iteration to each point $i \in [1, 100,000]$:

- $z_{i,1} = F(z_{i,0}) = z_{i,0} + p$

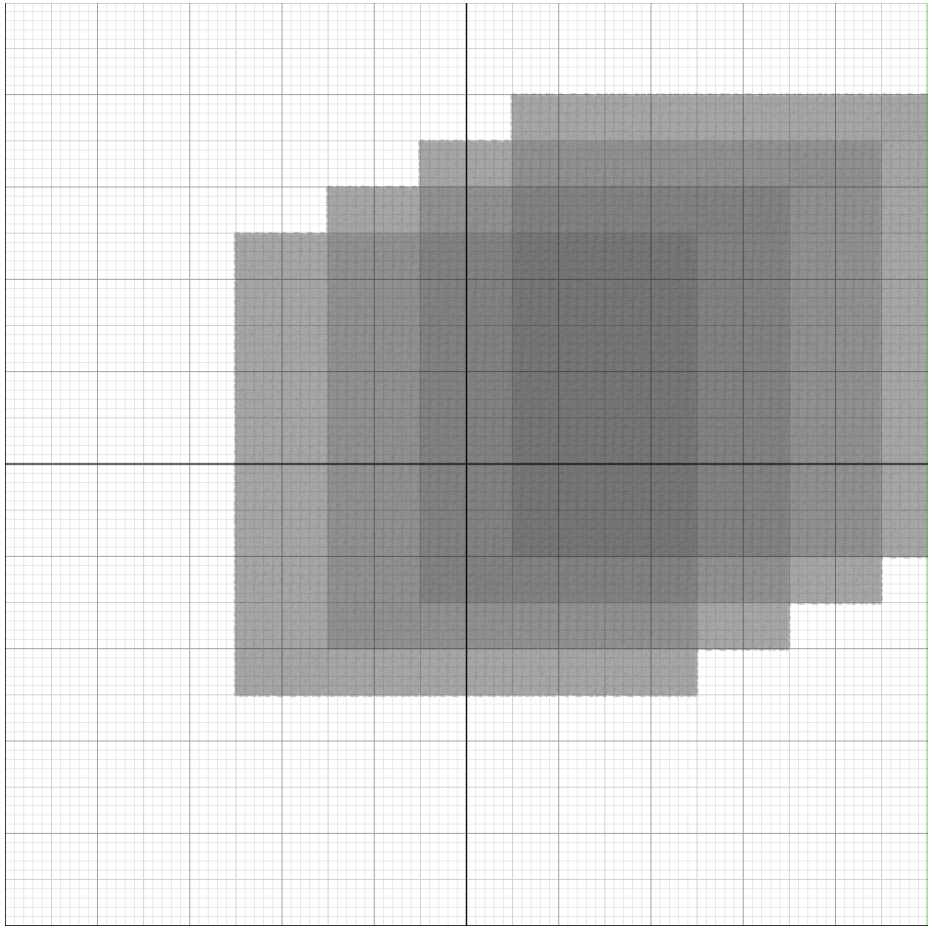
Translation: 100,000 points, 2 iteration



Apply the iteration to each point $i \in [1, 100,000]$:

- $z_{i,1} = F(z_{i,0}) = z_{i,0} + p$
- $z_{i,2} = F(z_{i,1}) = z_{i,1} + p$

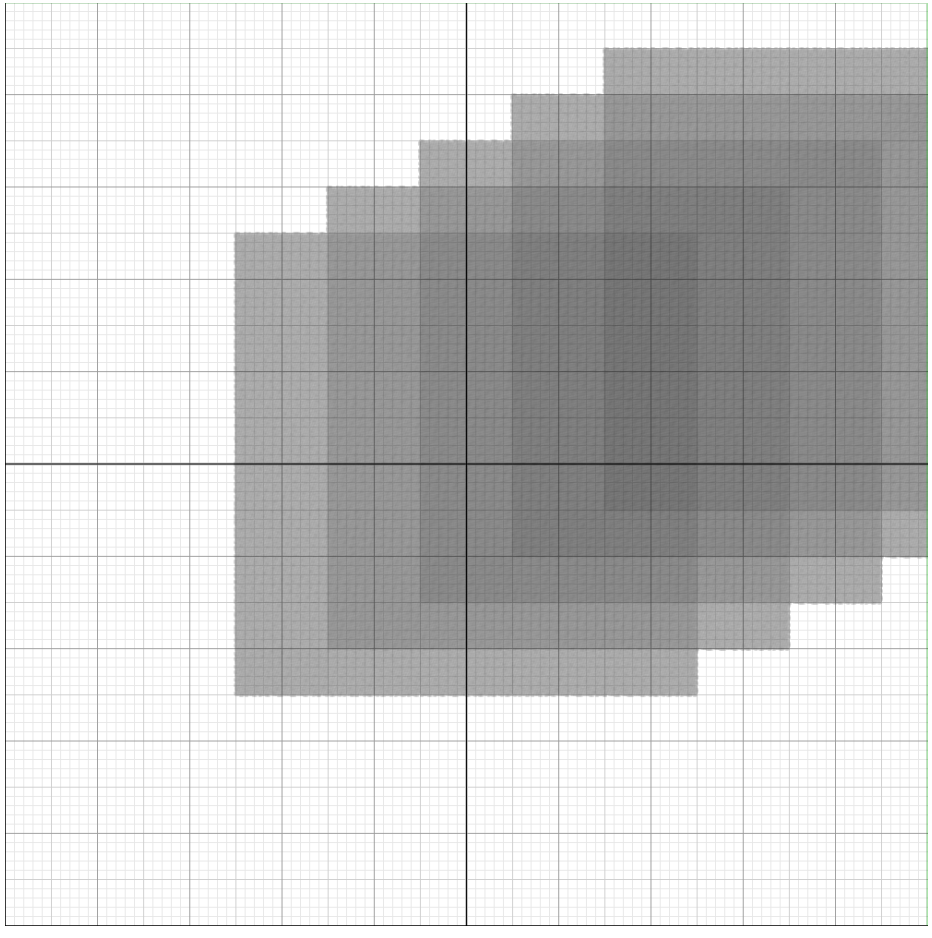
Translation: 100,000 points, 3 iteration



Apply the iteration to each point $i \in [1, 100,000]$:

- $z_{i,1} = F(z_{i,0}) = z_{i,0} + p$
- $z_{i,2} = F(z_{i,1}) = z_{i,1} + p$
- $z_{i,3} = F(z_{i,2}) = z_{i,2} + p$

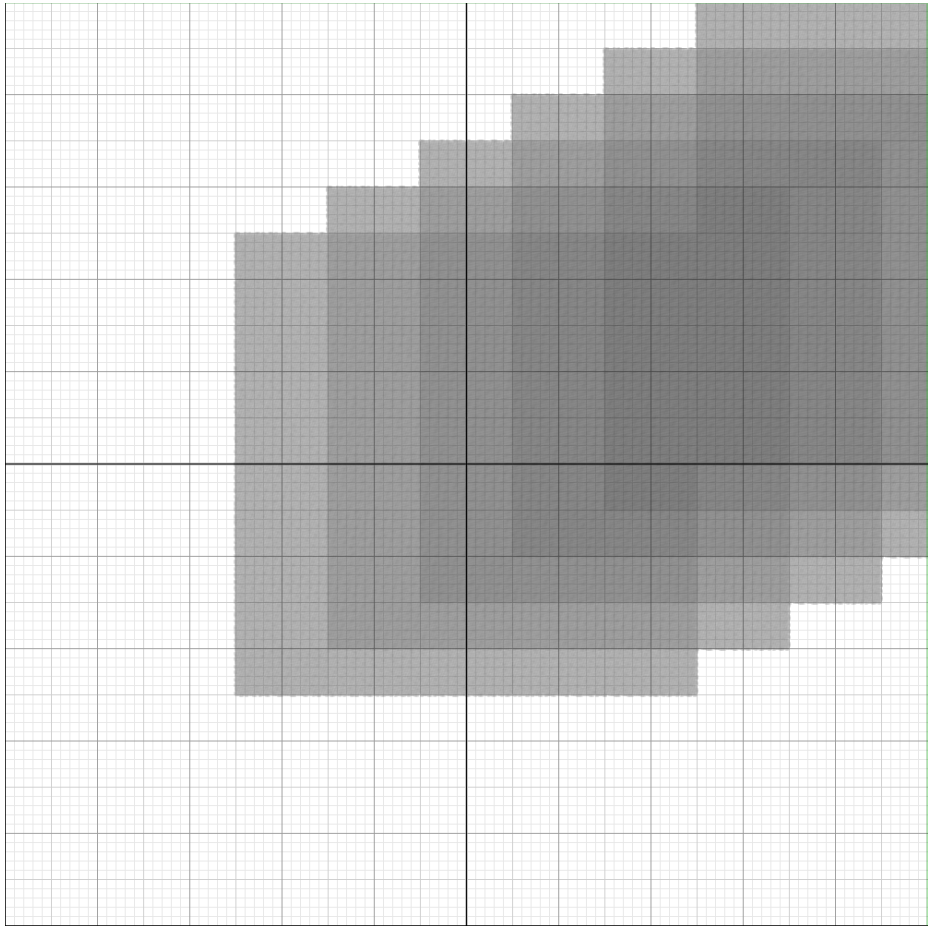
Translation: 100,000 points, 4 iteration



Apply the iteration to each point $i \in [1, 100,000]$:

- $z_{i,1} = F(z_{i,0}) = z_{i,0} + p$
- $z_{i,2} = F(z_{i,1}) = z_{i,1} + p$
- $z_{i,3} = F(z_{i,2}) = z_{i,2} + p$
- $z_{i,4} = F(z_{i,3}) = z_{i,3} + p$

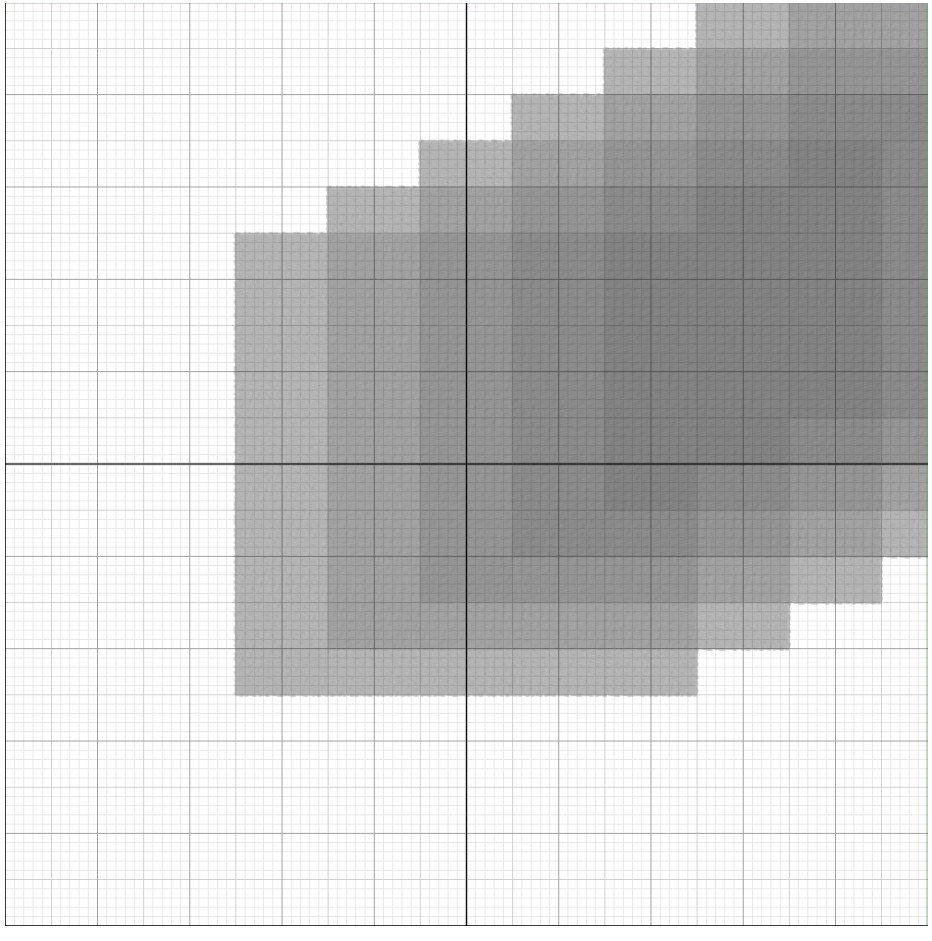
Translation: 100,000 points, 5 iteration



Apply the iteration to each point $i \in [1, 100,000]$:

- $z_{i,1} = F(z_{i,0}) = z_{i,0} + p$
- $z_{i,2} = F(z_{i,1}) = z_{i,1} + p$
- $z_{i,3} = F(z_{i,2}) = z_{i,2} + p$
- $z_{i,4} = F(z_{i,3}) = z_{i,3} + p$
- $z_{i,5} = F(z_{i,4}) = z_{i,4} + p$

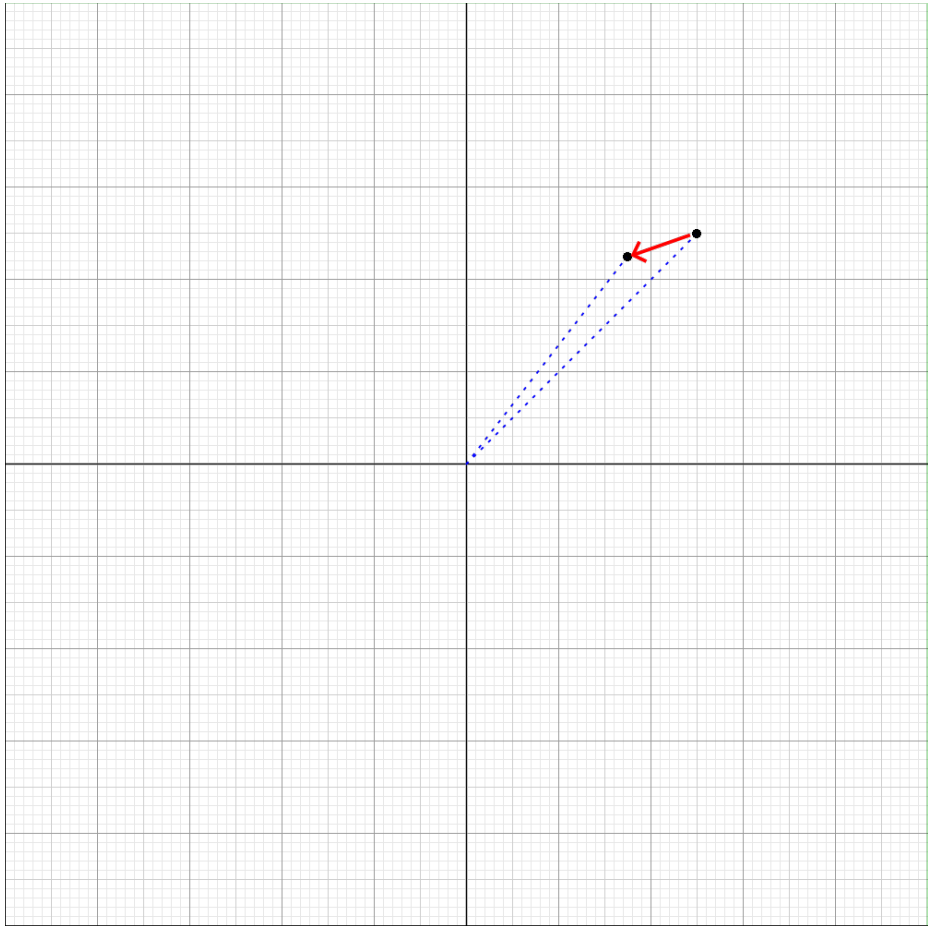
Translation: 100,000 points, 6 iteration



Apply the iteration to each point $i \in [1, 100,000]$:

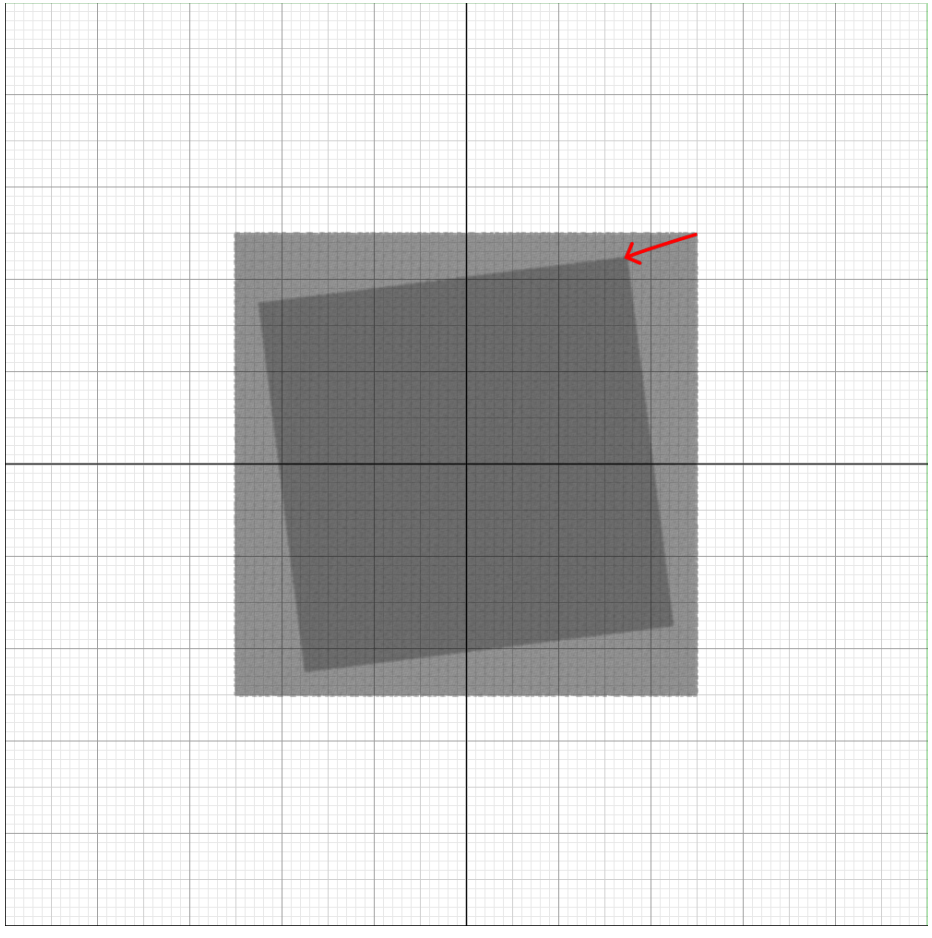
- $z_{i,1} = F(z_{i,0}) = z_{i,0} + p$
- $z_{i,2} = F(z_{i,1}) = z_{i,1} + p$
- $z_{i,3} = F(z_{i,2}) = z_{i,2} + p$
- $z_{i,4} = F(z_{i,3}) = z_{i,3} + p$
- $z_{i,5} = F(z_{i,4}) = z_{i,4} + p$
- $z_{i,6} = F(z_{i,5}) = z_{i,5} + p$

Another Iterations: Scaling



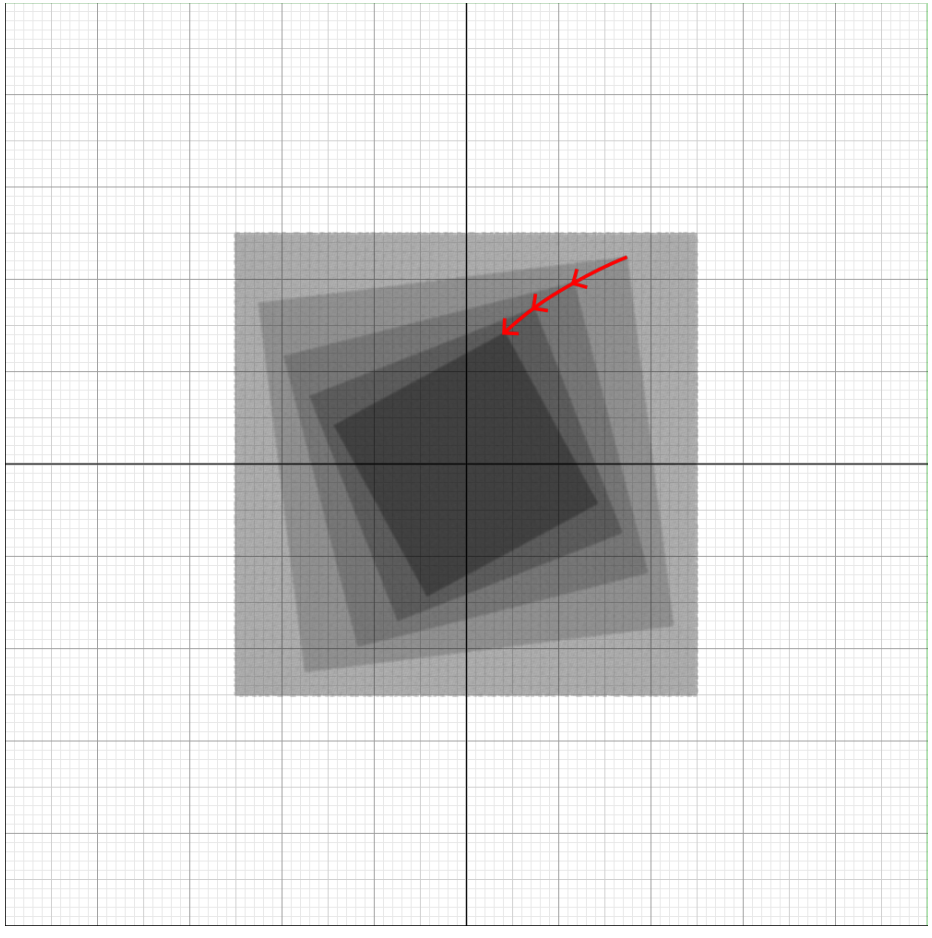
- $z_{n+1} = S \cdot z_n$
- S - complex scaling parameter
-

Scaling: 1 iteration



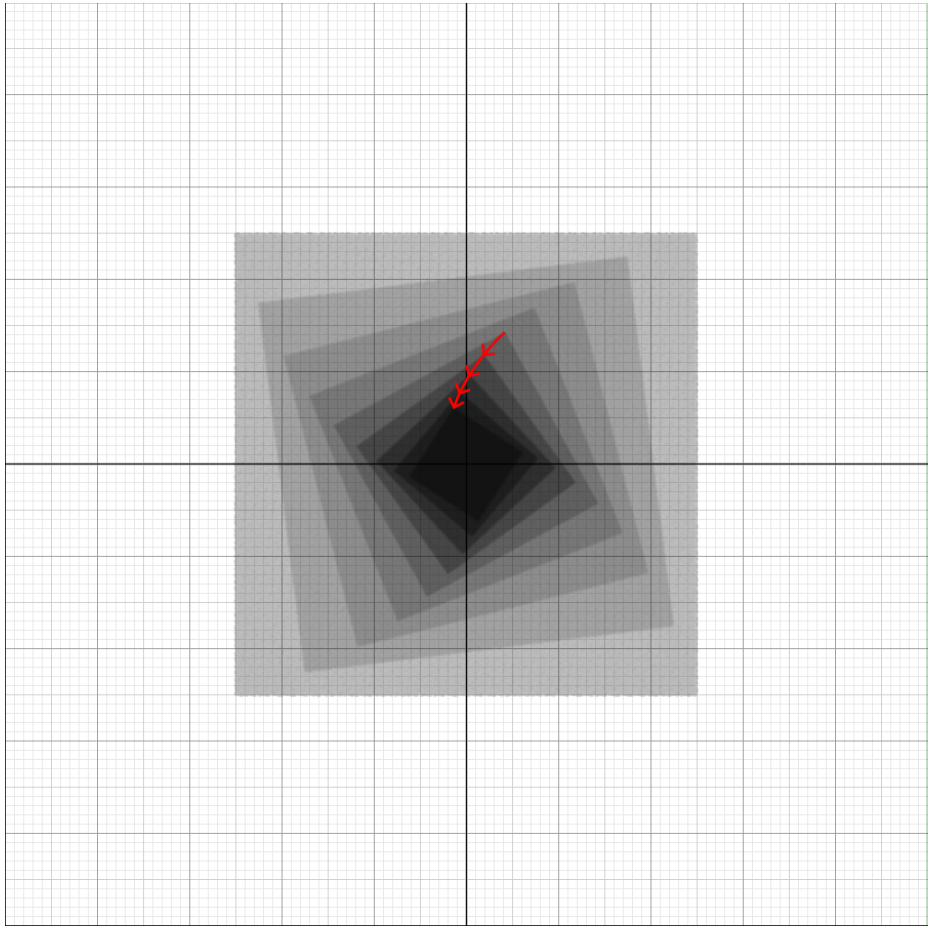
- $z_{n+1} = S \cdot z_n$
- S - complex scaling parameter
- 1 iteration
-

Scaling: 4 iteration



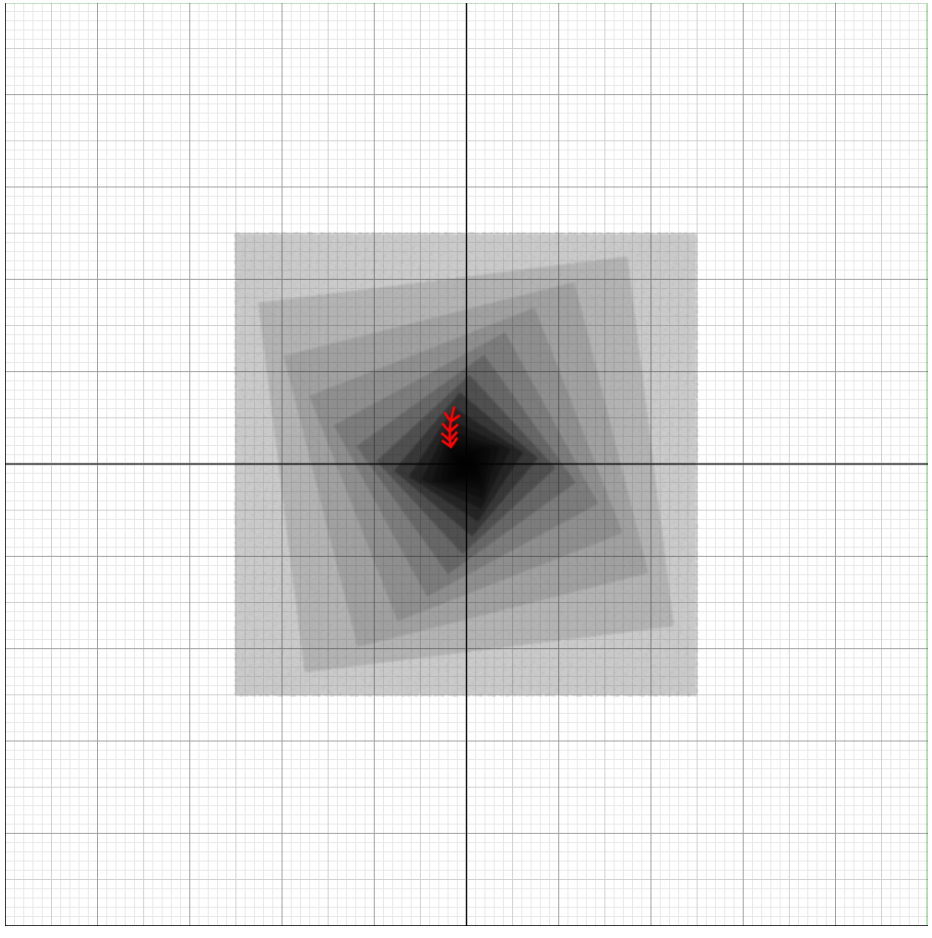
- $z_{n+1} = S \cdot z_n$
- S - complex scaling parameter
- 4 iteration
-

Scaling: 8 iteration



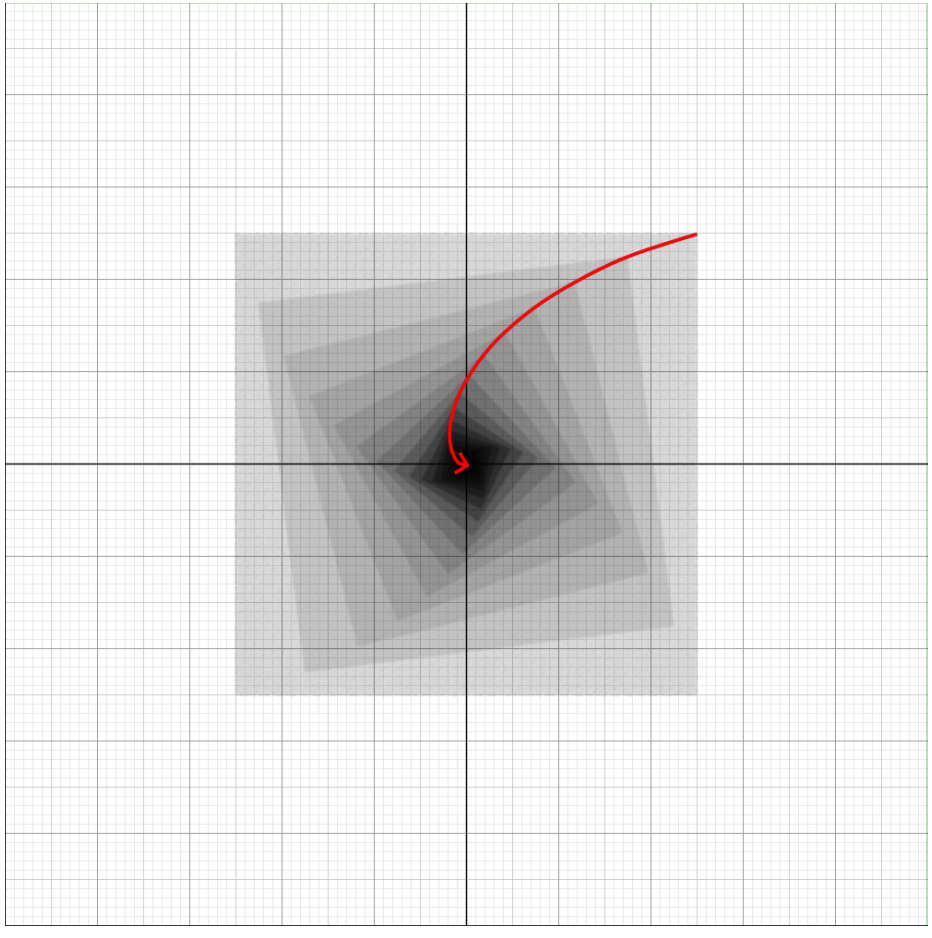
- $z_{n+1} = S \cdot z_n$
- S - complex scaling parameter
- 8 iteration
-

Scaling: 16 iteration



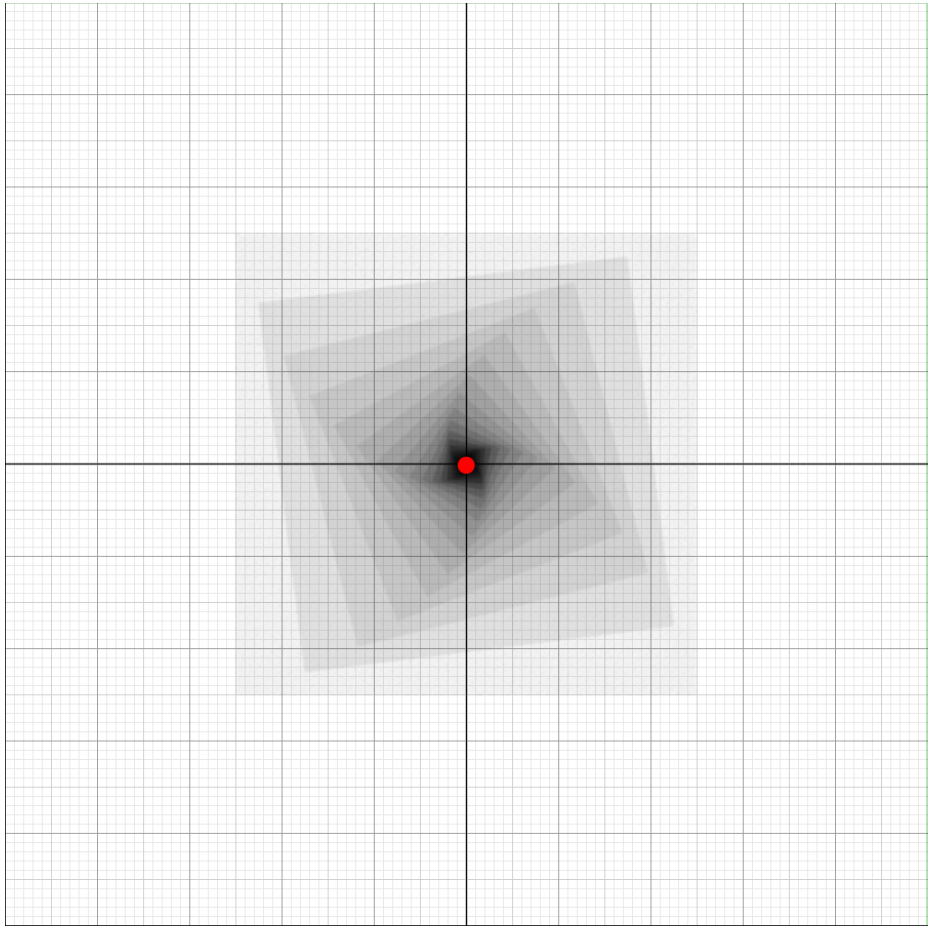
- $z_{n+1} = S \cdot z_n$
- S - complex scaling parameter
- 16 iteration
-

Scaling: 32 iteration



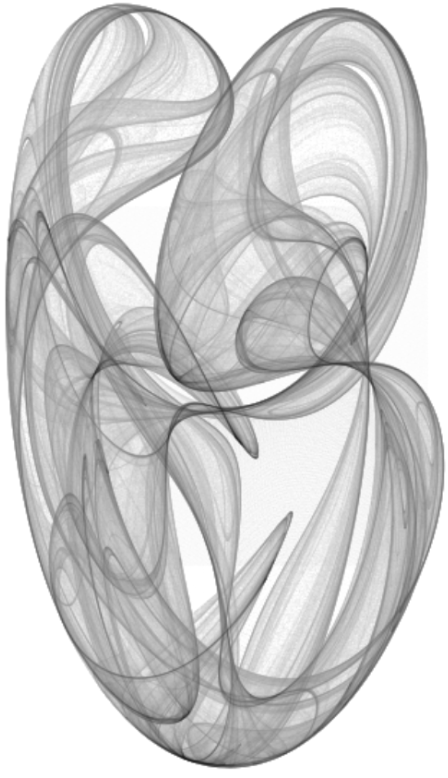
- $z_{n+1} = S \cdot z_n$
- S - complex scaling parameter
- 32 iteration
-

Limit Point: Attractor!



- $z_{n+1} = S \cdot z_n$
- S - complex scaling parameter
- Limit of the scaling iterations is a single point
- It is called **Attractor**
-

Clifford Attractor



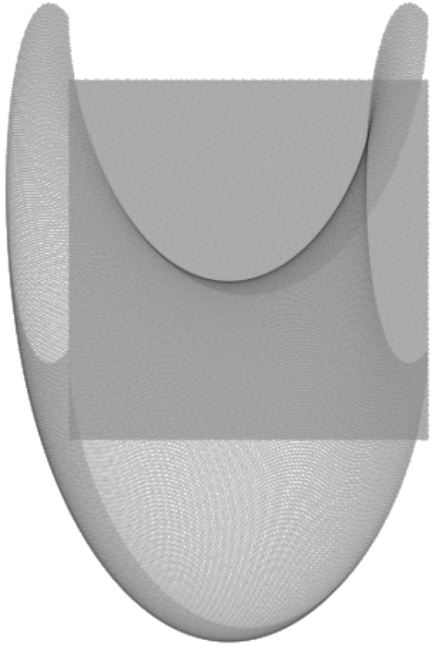
- $x_{n+1} = \sin(a \cdot y_n) + c \cos(a \cdot x_n)$
- $y_{n+1} = \sin(b \cdot x_n) + d \cos(b \cdot y_n)$
- a, b, c, d - arbitrary parameters.

The attractor exhibits rather complex chaotic behavior.

It was first described in 1990 book

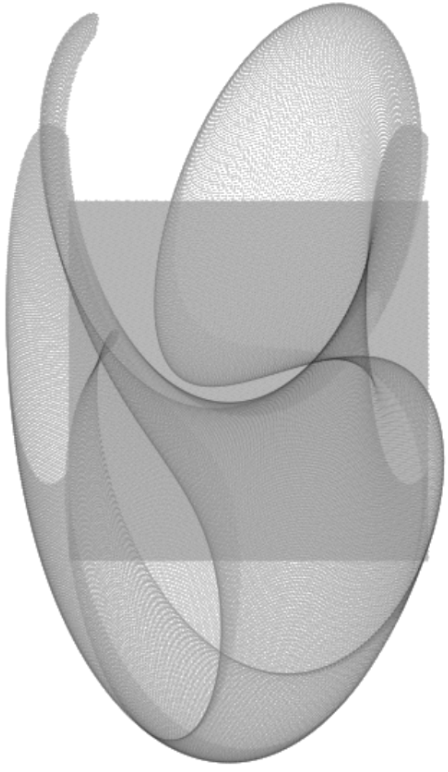
Clifford Pickover 'Chaos in Wonderland'

Clifford Attractor I



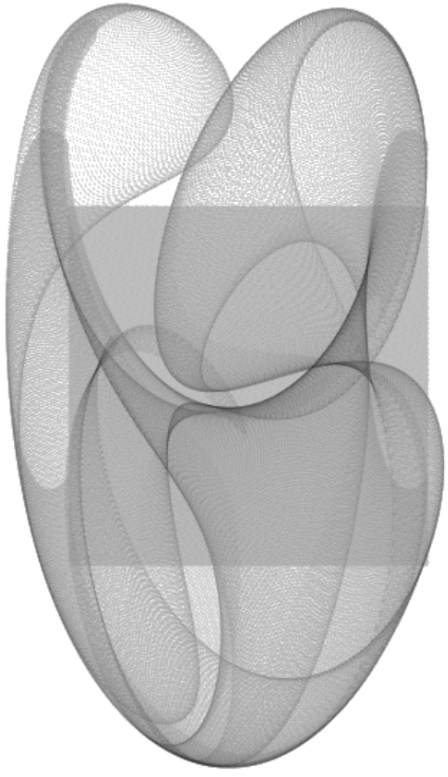
- $x_{n+1} = \sin(a \cdot y_n) + c \cos(a \cdot x_n)$
- $y_{n+1} = \sin(b \cdot x_n) + d \cos(b \cdot y_n)$
- 1 iteration

Clifford Attractor 2



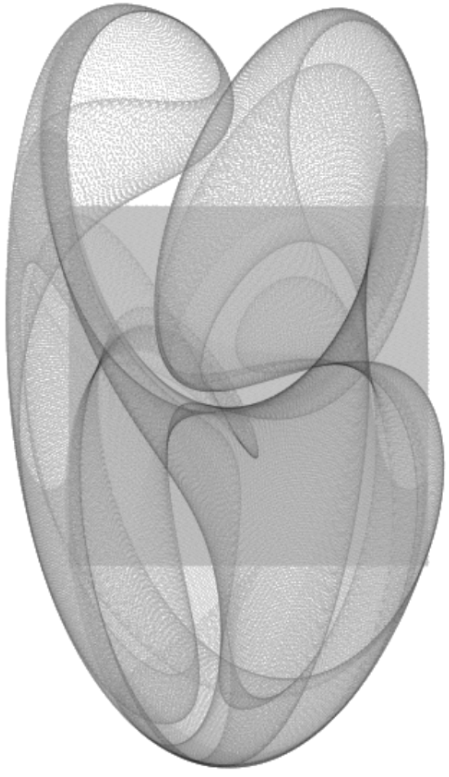
- $x_{n+1} = \sin(a \cdot y_n) + c \cos(a \cdot x_n)$
- $y_{n+1} = \sin(b \cdot x_n) + d \cos(b \cdot y_n)$
- 2 iterations

Clifford Attractor 3



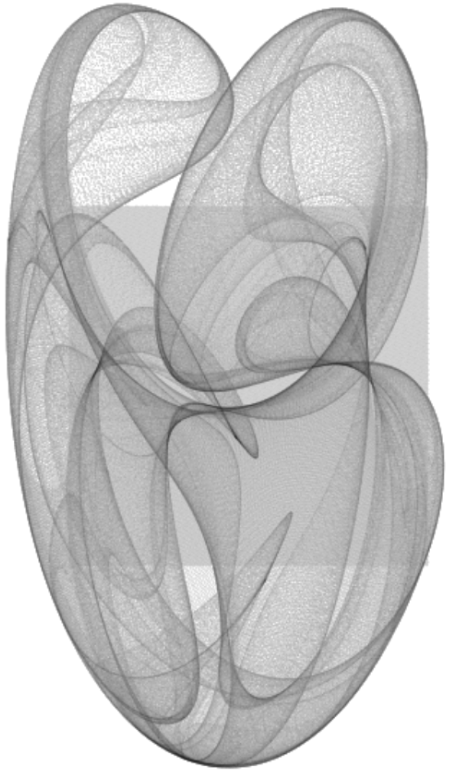
- $x_{n+1} = \sin(a \cdot y_n) + c \cos(a \cdot x_n)$
- $y_{n+1} = \sin(b \cdot x_n) + d \cos(b \cdot y_n)$
- 3 iterations

Clifford Attractor 4



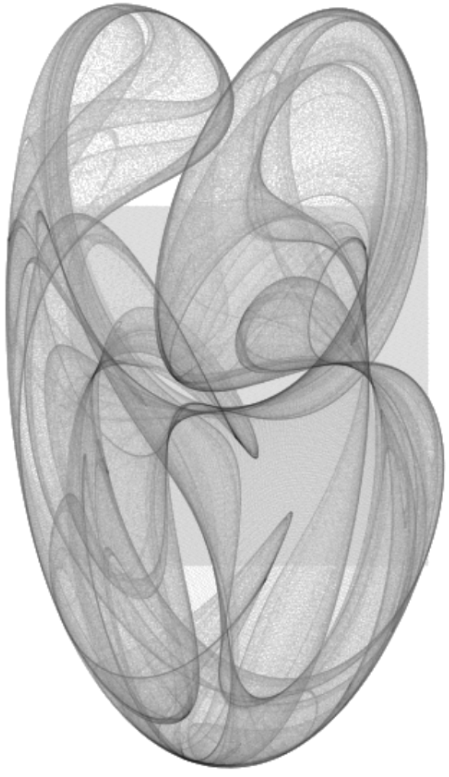
- $x_{n+1} = \sin(a \cdot y_n) + c \cos(a \cdot x_n)$
- $y_{n+1} = \sin(b \cdot x_n) + d \cos(b \cdot y_n)$
- 4 iterations

Clifford Attractor 8



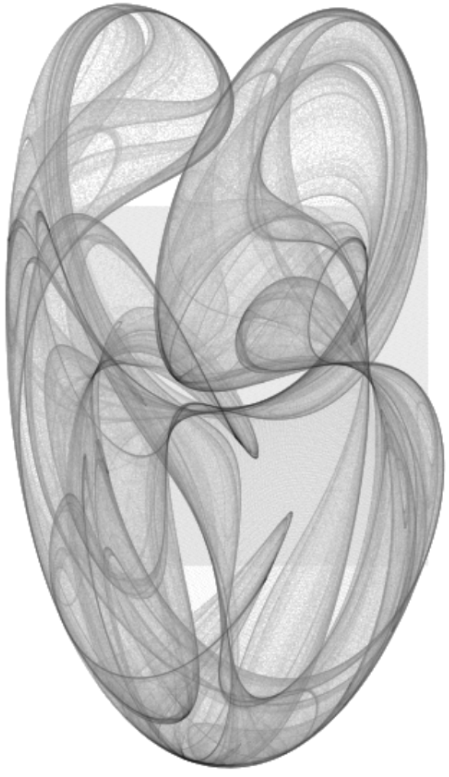
- $x_{n+1} = \sin(a \cdot y_n) + c \cos(a \cdot x_n)$
- $y_{n+1} = \sin(b \cdot x_n) + d \cos(b \cdot y_n)$
- 8 iterations

Clifford Attractor 16



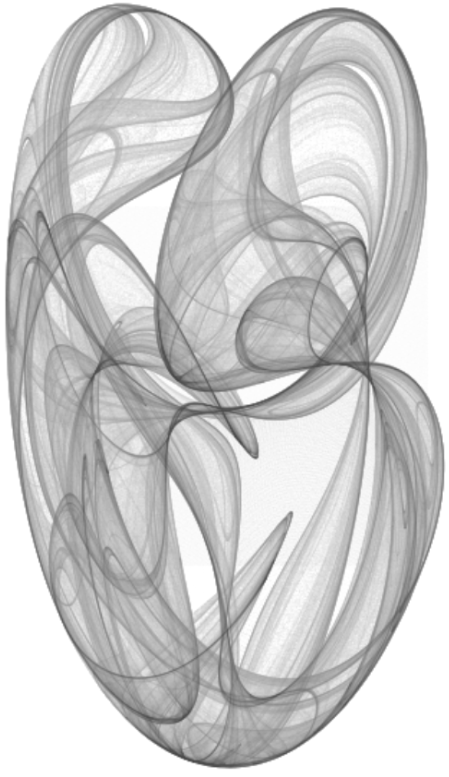
- $x_{n+1} = \sin(a \cdot y_n) + c \cos(a \cdot x_n)$
- $y_{n+1} = \sin(b \cdot x_n) + d \cos(b \cdot y_n)$
- 16 iterations

Clifford Attractor 32



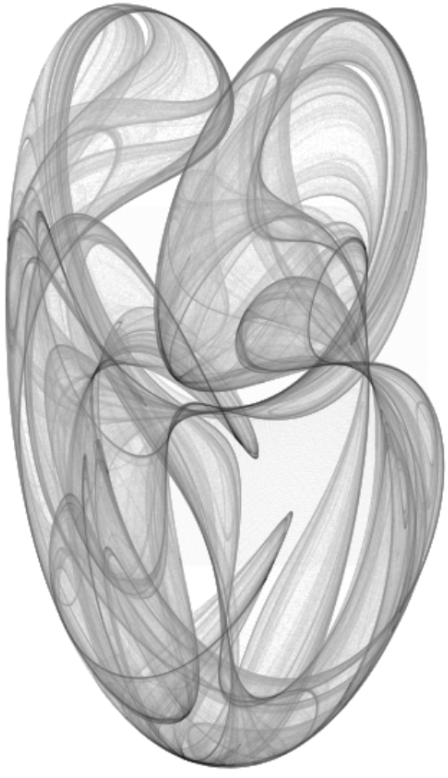
- $x_{n+1} = \sin(a \cdot y_n) + c \cos(a \cdot x_n)$
- $y_{n+1} = \sin(b \cdot x_n) + d \cos(b \cdot y_n)$
- 32 iterations

Clifford Attractor 64



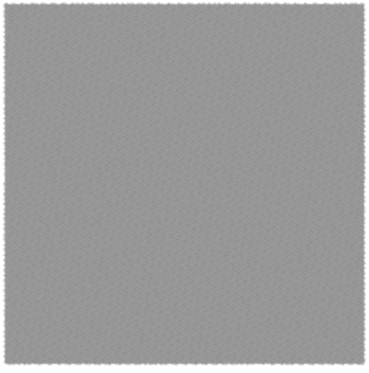
- $x_{n+1} = \sin(a \cdot y_n) + c \cos(a \cdot x_n)$
- $y_{n+1} = \sin(b \cdot x_n) + d \cos(b \cdot y_n)$
- 64 iterations

Clifford Attractor 128



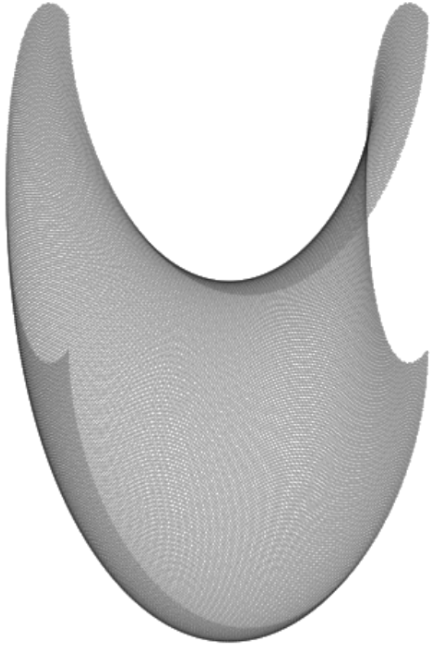
- $x_{n+1} = \sin(a \cdot y_n) + c \cos(a \cdot x_n)$
- $y_{n+1} = \sin(b \cdot x_n) + d \cos(b \cdot y_n)$
- 128 iterations

Clifford Attractor Convergence #0



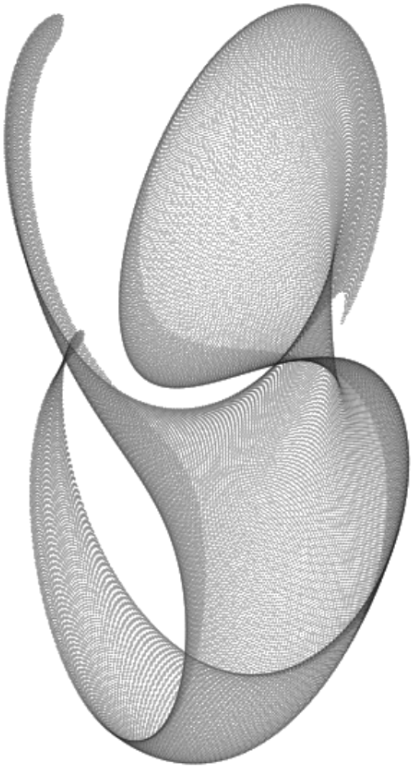
- Initial points

Clifford Attractor Convergence #1



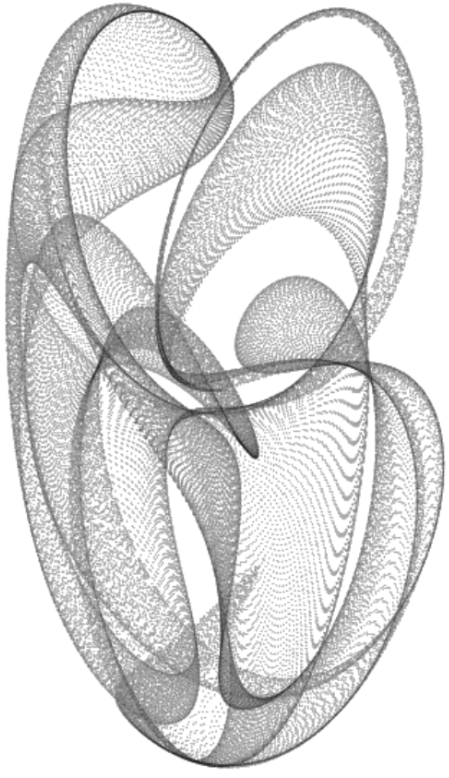
- iteration #1

Clifford Attractor Convergence #2



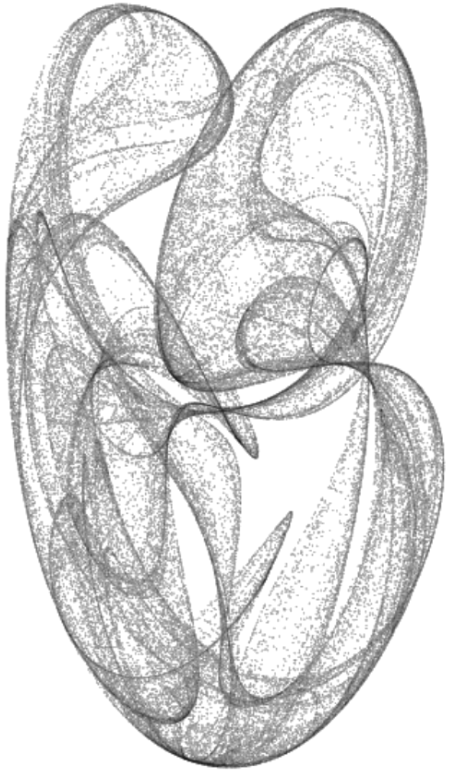
- iteration #2

Clifford Attractor Convergence #4



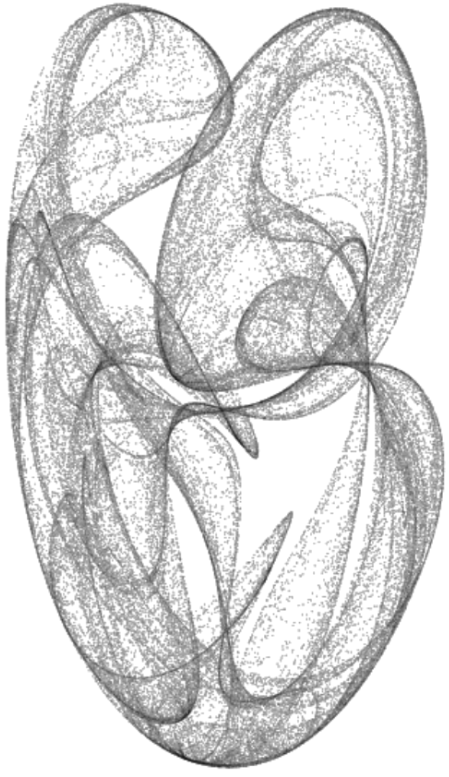
- iteration #4

Clifford Attractor Convergence #16



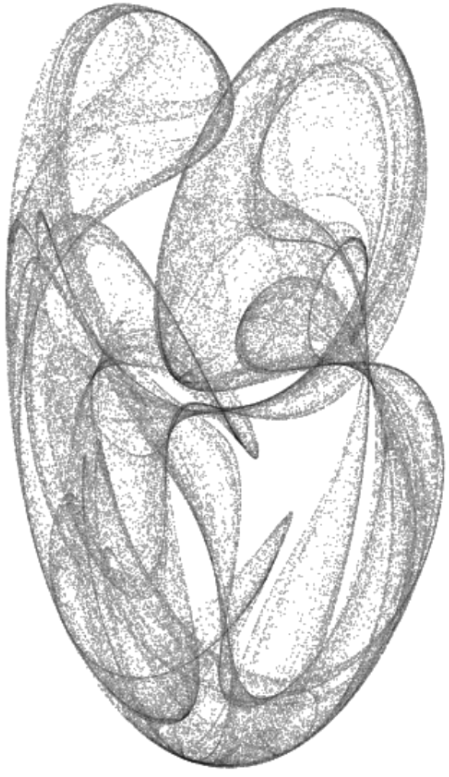
- iteration #16
- No visible changes.
- Iterations became chaotic.

Clifford Attractor Convergence #17



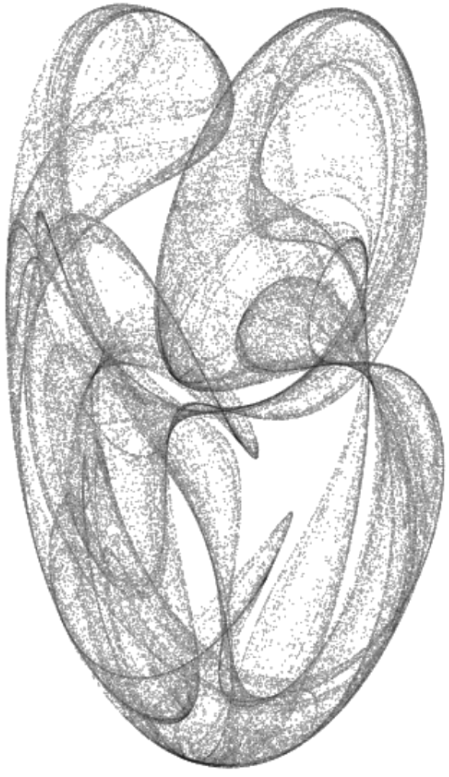
- iteration #17
- No visible changes.
- Iterations became chaotic.

Clifford Attractor Convergence #18



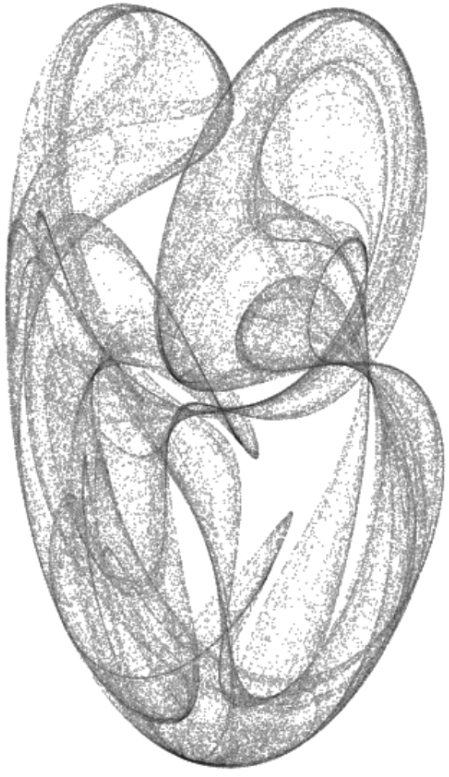
- iteration #18
- No visible changes.
- Iterations became chaotic.

Clifford Attractor Convergence #19



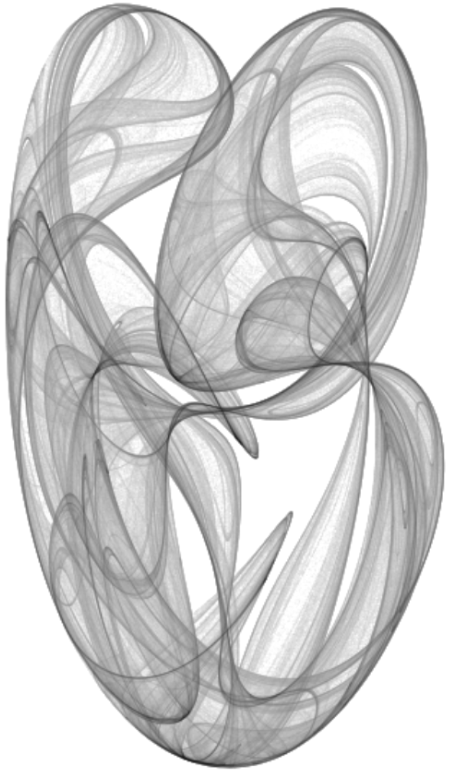
- iteration #19
- No visible changes.
- Iterations became chaotic.

Clifford Attractor Convergence #20



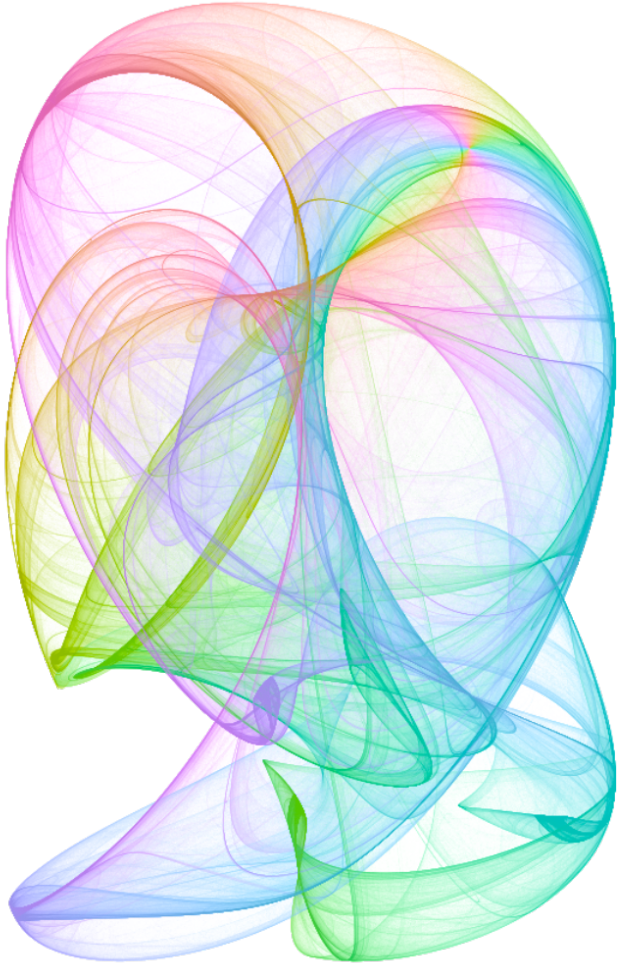
- iteration #20
- No visible changes.
- Iterations became chaotic.

Clifford Attractor Convergence #16 to #100

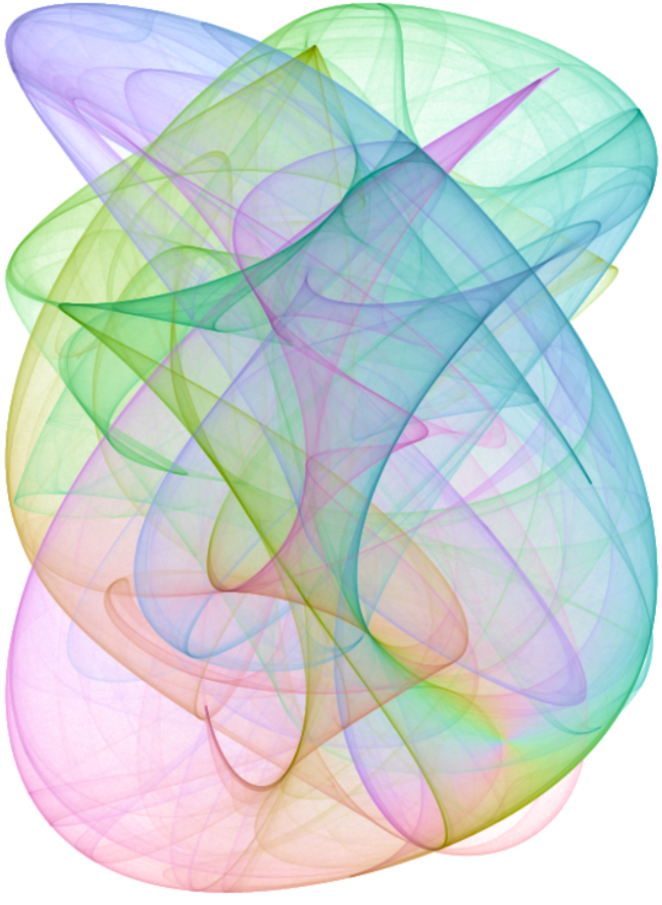


- Average of all chaotic iterations
- from #16 to #100
- Approximately 10,000,000 points

Clifford Attractor Examples I



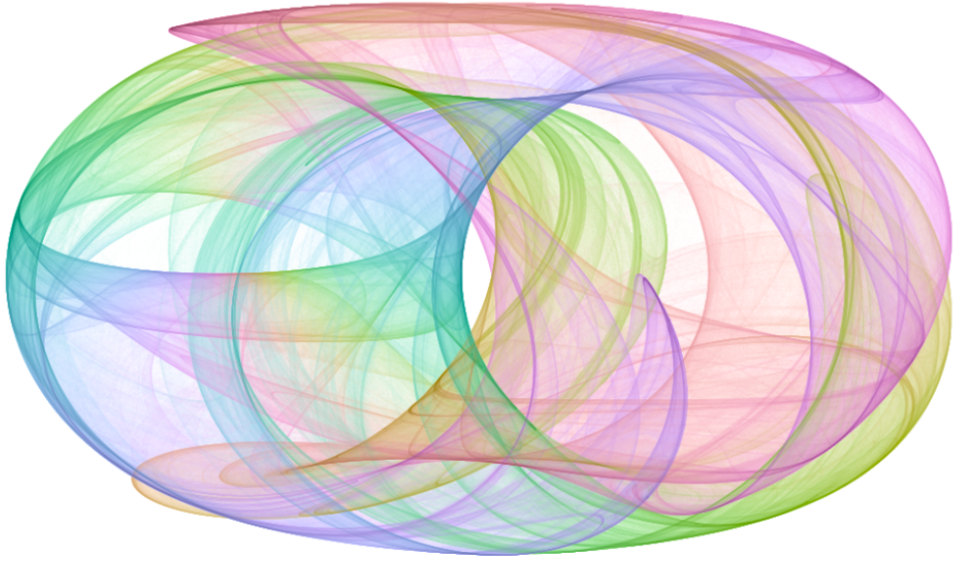
Clifford Attractor Examples 2



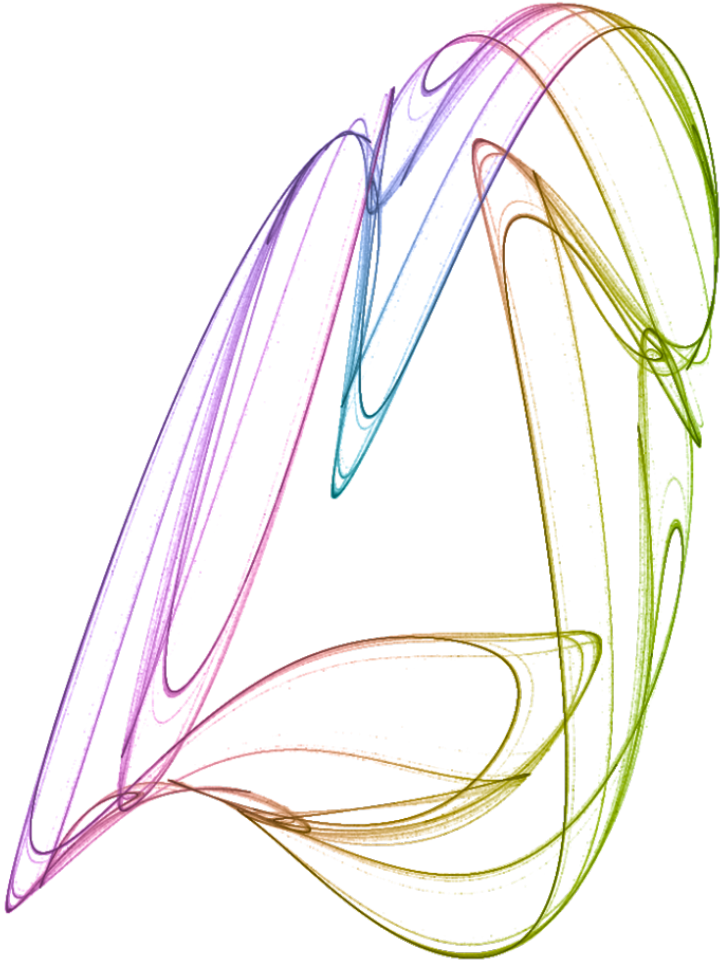
Clifford Attractor Examples 3



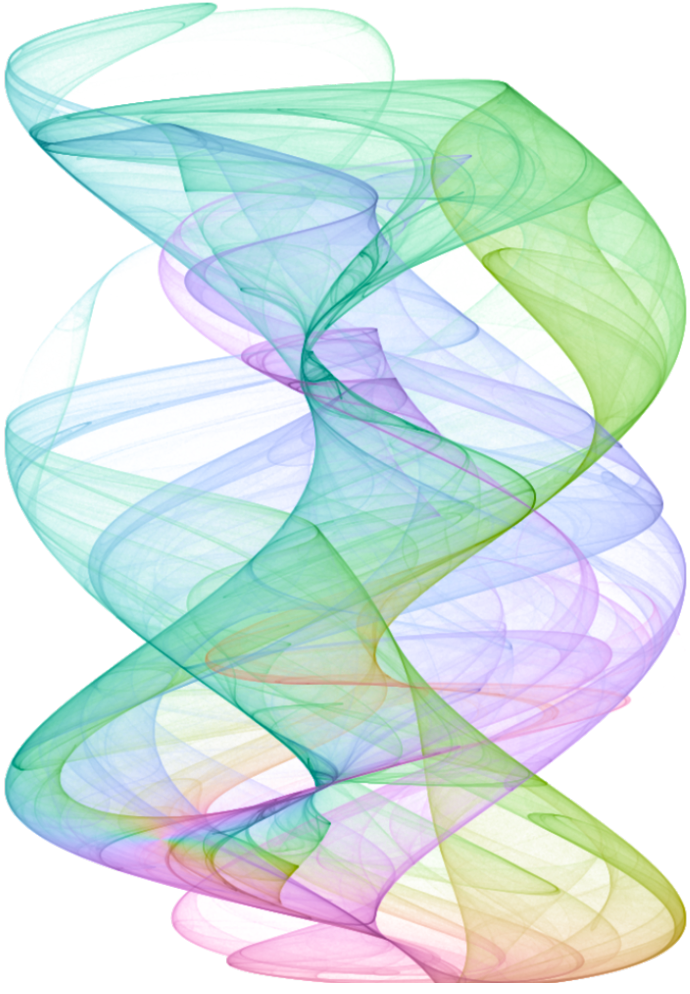
Clifford Attractor Examples 4



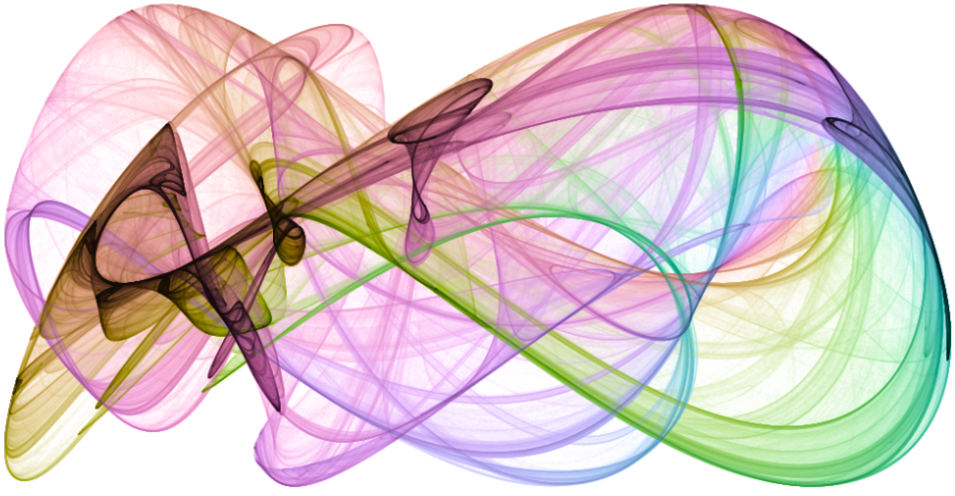
Clifford Attractor Examples 5



Clifford Attractor Examples 6



Clifford Attractor Examples 7

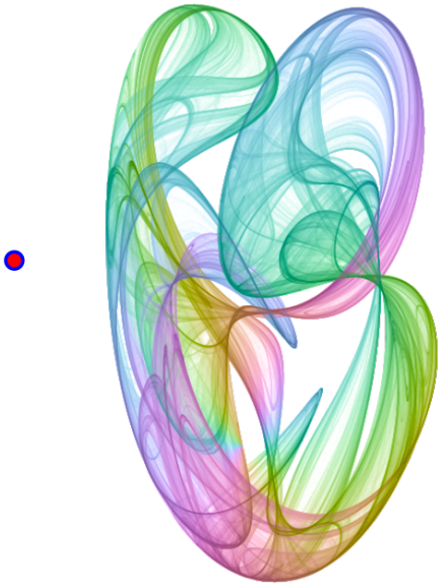


Making Attractor Symmetric



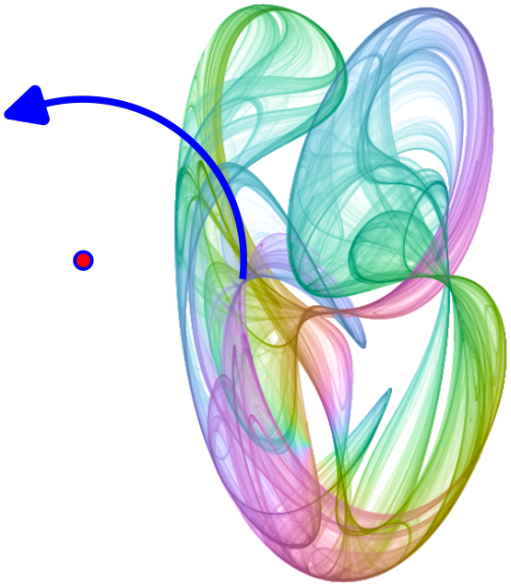
- Lets take attractor with selected parameters

Making Attractor Symmetric 2



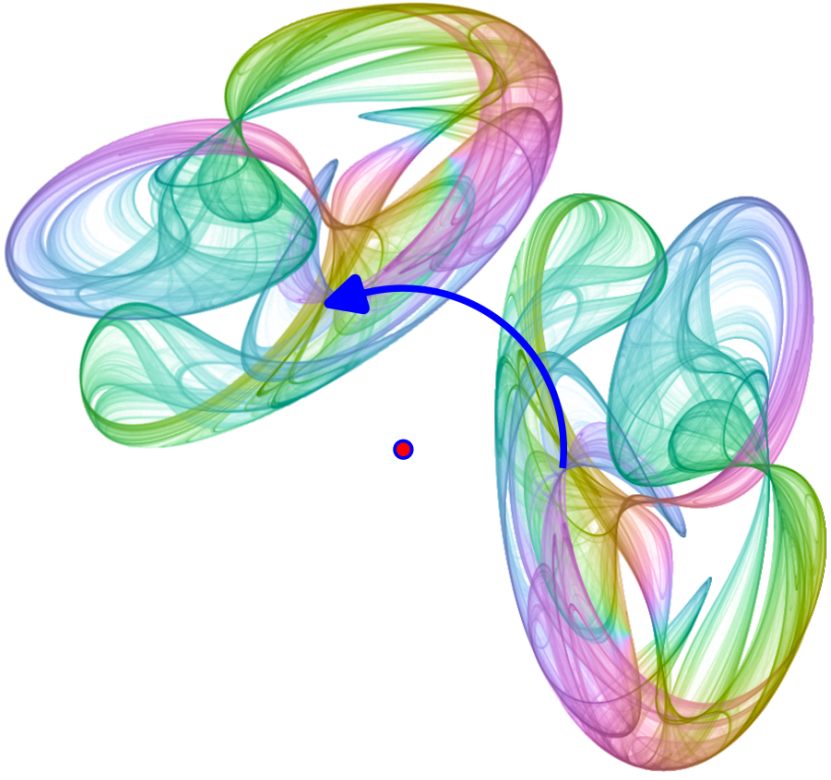
- Lets take attractor with selected parameters
- Select arbitrary center of rotations

Making Attractor Symmetric 3



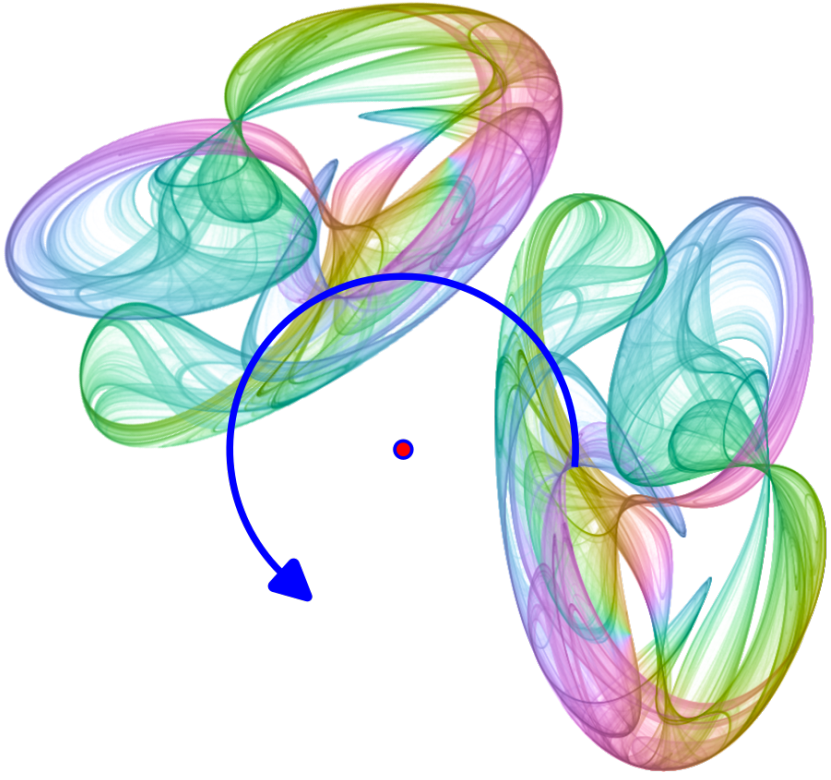
- Lets take attractor with selected parameters
- Select arbitrary center of rotations
- Apply counterclockwise rotation $\frac{2\pi}{3}$

Making Attractor Symmetric 4



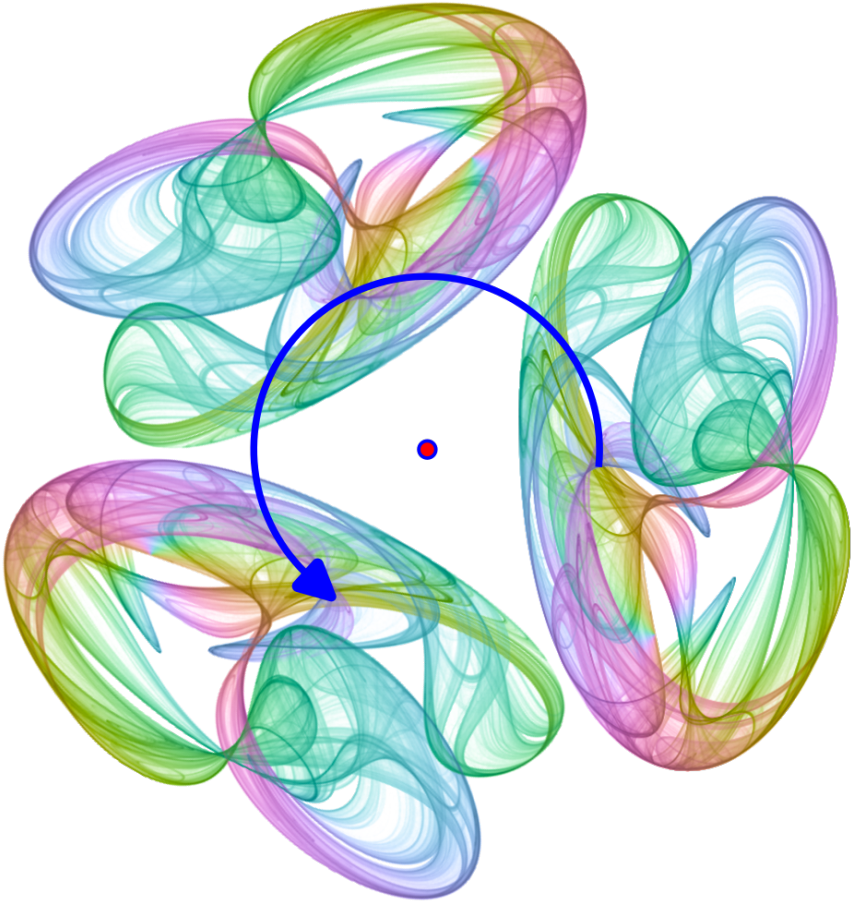
- Lets take attractor with selected parameters
- Select arbitrary center of rotations
- Apply counterclockwise rotation $\frac{2\pi}{3}$

Making Attractor Symmetric 5



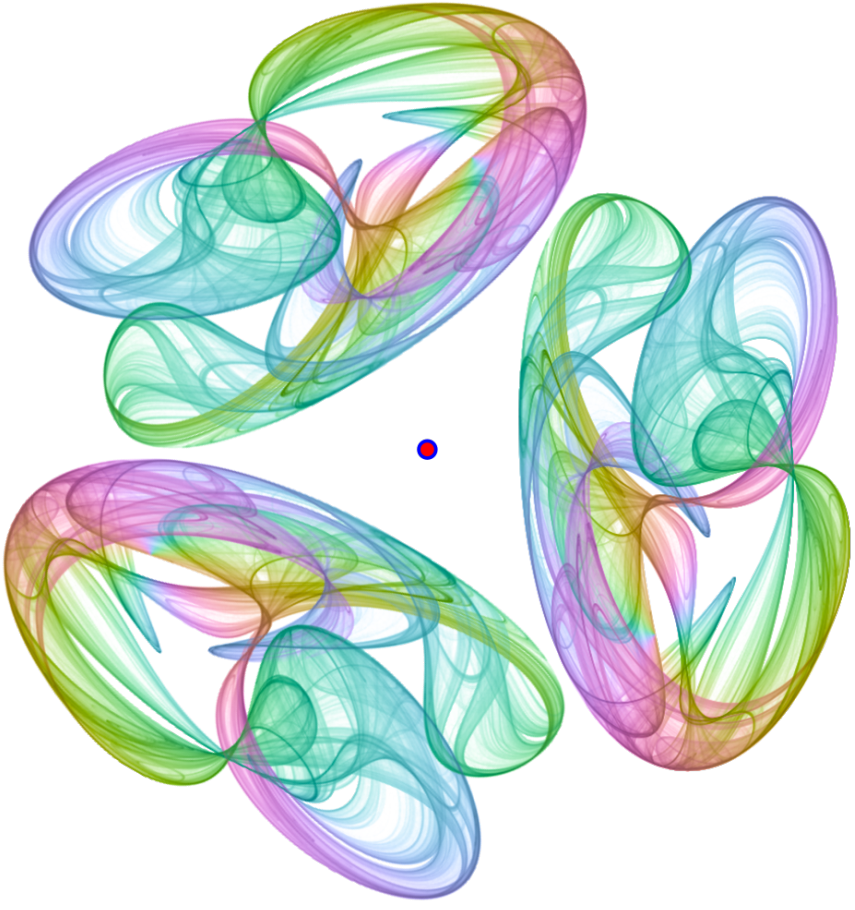
- Lets take attractor with selected parameters
- Select arbitrary center of rotations
- Apply counterclockwise rotation $\frac{2\pi}{3}$
- Apply counterclockwise rotation $\frac{4\pi}{3}$

Making Attractor Symmetric 6



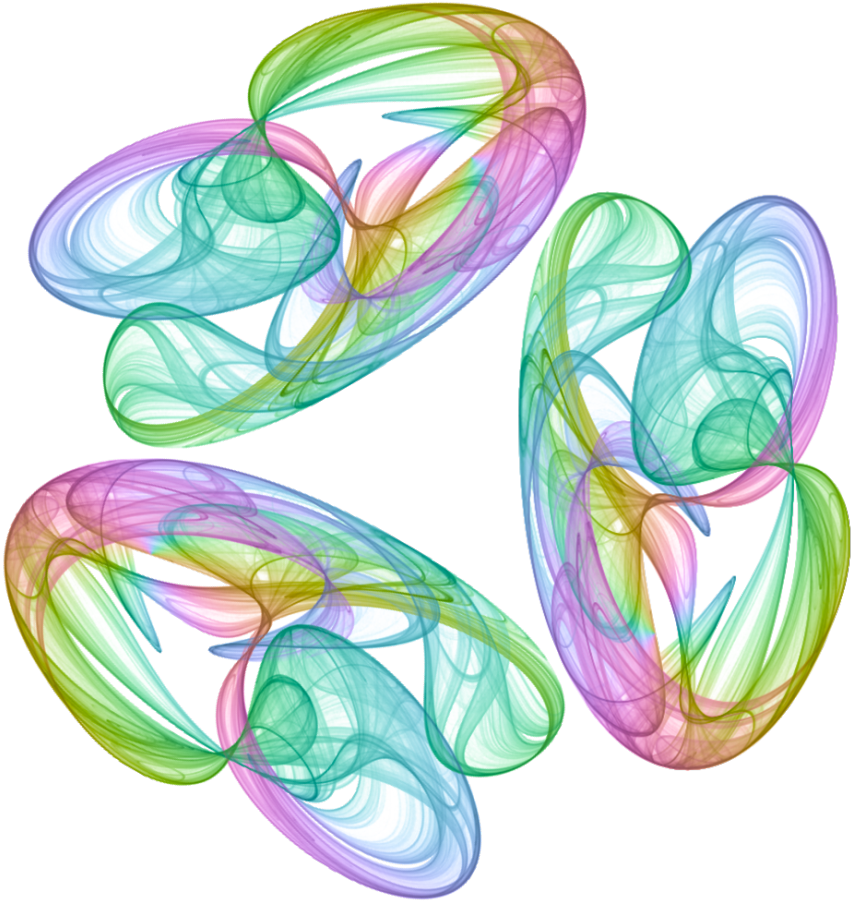
- Lets take attractor with selected parameters
- Select arbitrary center of rotations
- Apply counterclockwise rotation $\frac{2\pi}{3}$
- Apply counterclockwise rotation $\frac{4\pi}{3}$

Making Attractor Symmetric 7



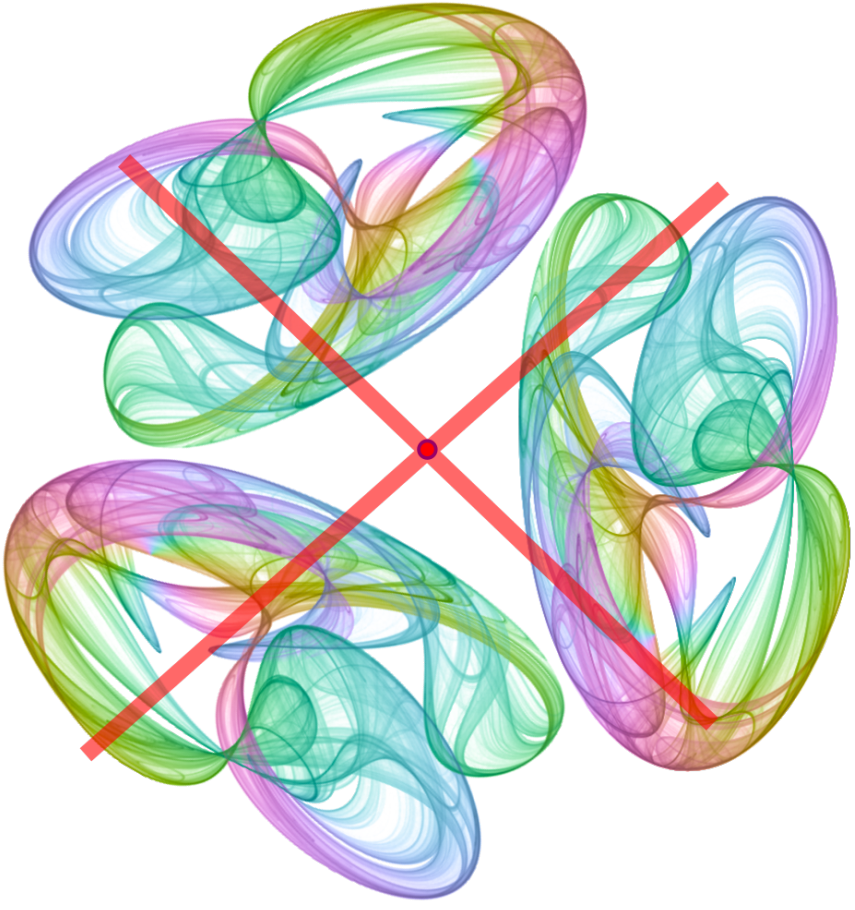
- We've got **"attractor"** with tri-fold symmetry

Transparencies



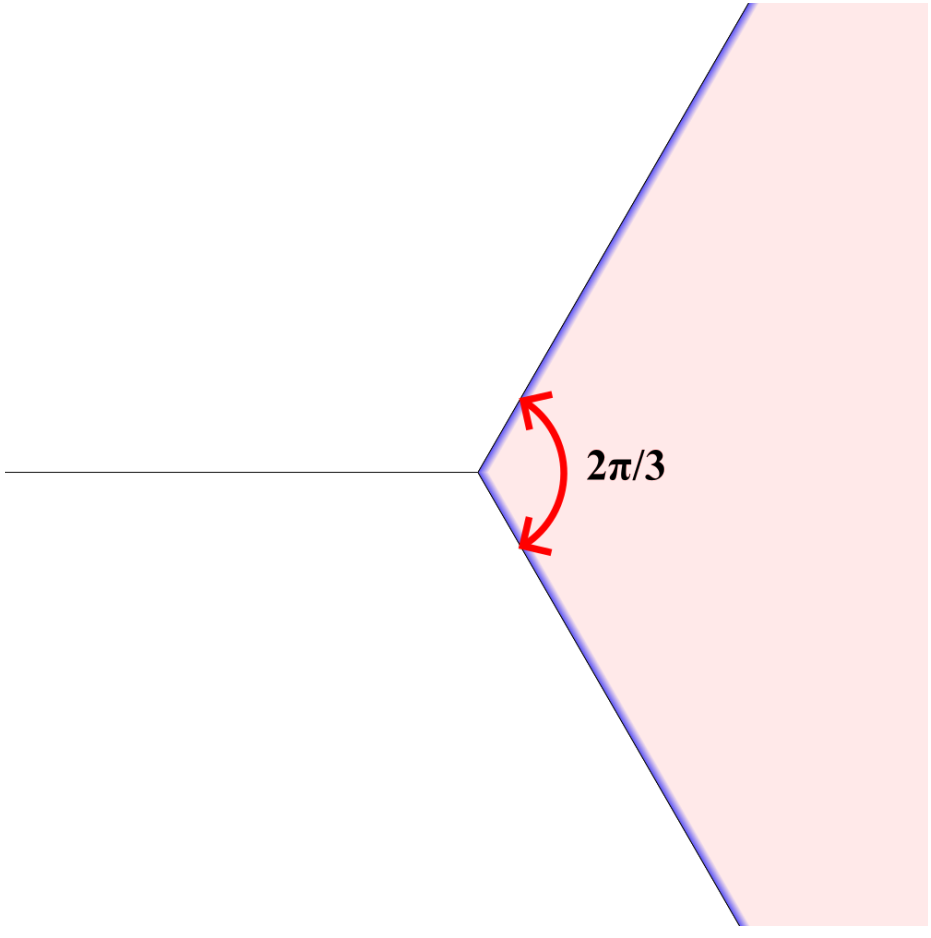
- This construction is rather simple and un-interesting
- It is similar to a stacking of several transparencies

Transparencies 2



- The transparencies method works universally for any images
- But the symmetry has no actual effect on the attractor
- We want to make the **attractor** to 'feel' the symmetry

Orbifold



Let's constrain iterations to the orbifold of the symmetry group

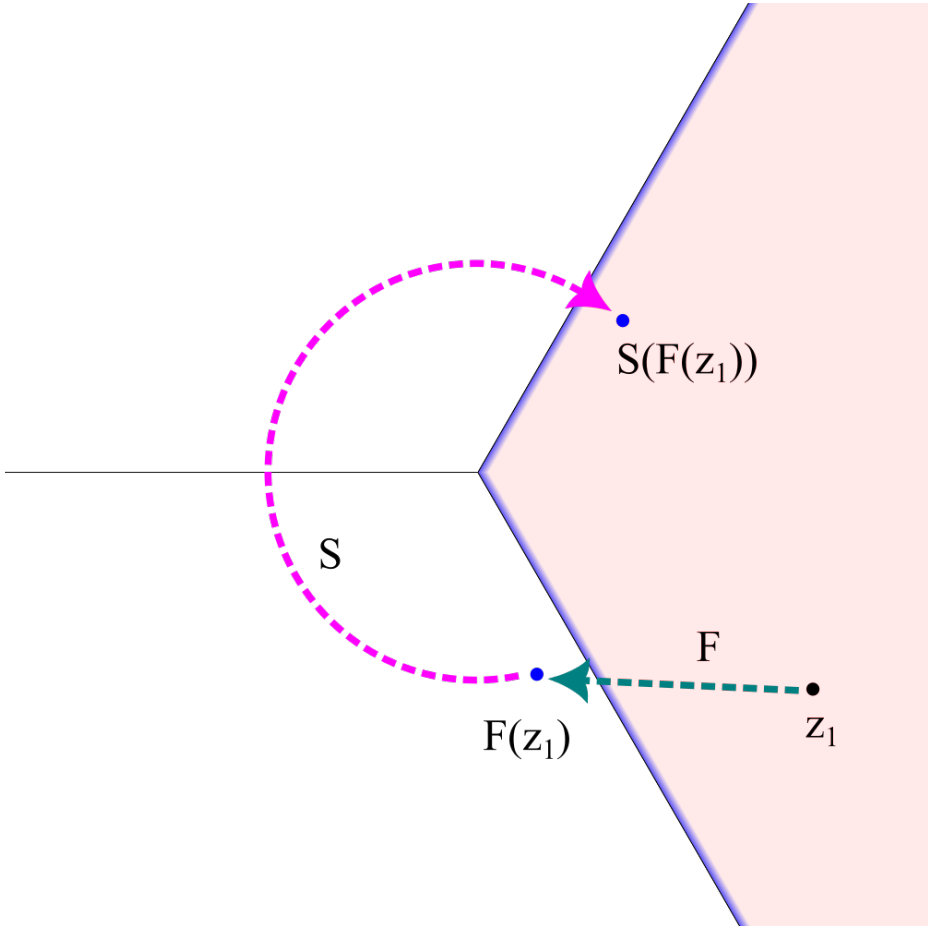
The tri-fold symmetry orbifold is the surface of an infinite cone.

The cone surface cut along straight line becomes infinite wedge with angle $\frac{2\pi}{3}$ at the apex (pink region)

The copies of orbifold image tile the plane

The unfolded orbifold is also called **fundamental domain**

Dynamics on Orbifold



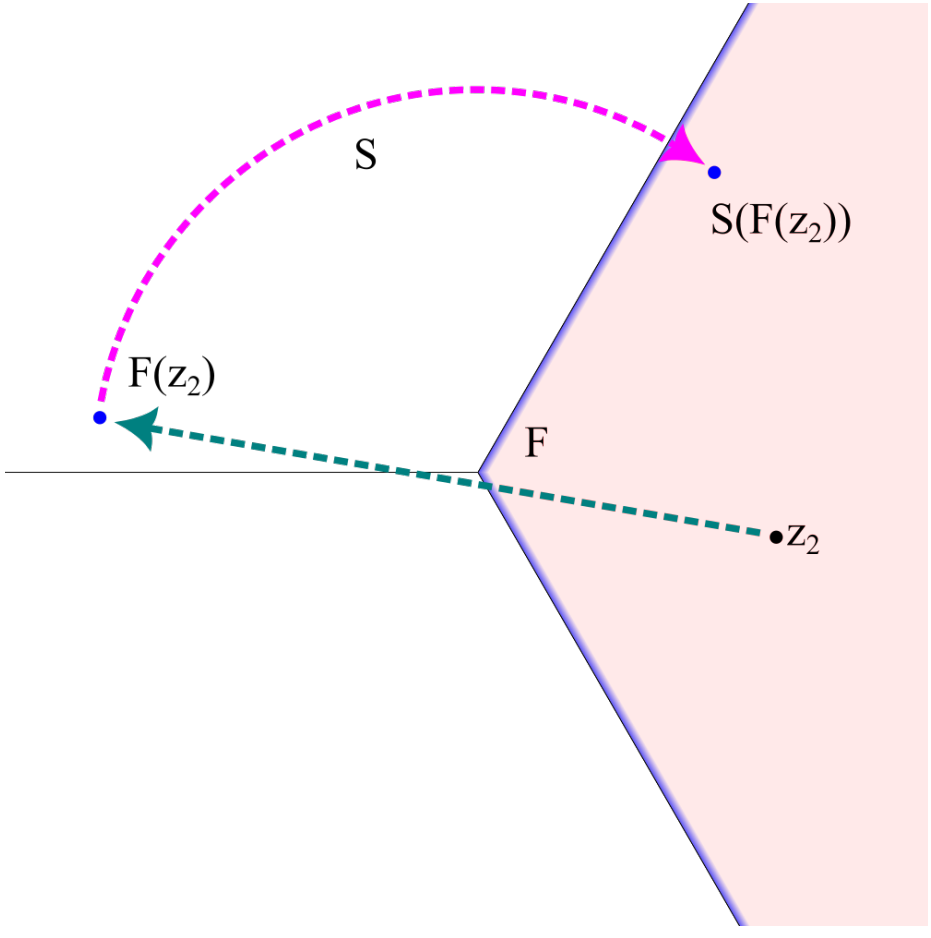
How to restrict iterations to the orbifold interior?

- Each iteration
- $z_{n+1} = F(z_n)$
- is composed with symmetrization function
  $z_{n+1} = S(F(z_n))$

Symmertization $S(z)$ moves any point on the plane into the corresponding point in the interior of the fundamental domain.

In this case it is clockwise rotation $\frac{4\pi}{3}$.

Dynamics on Orbifold 2



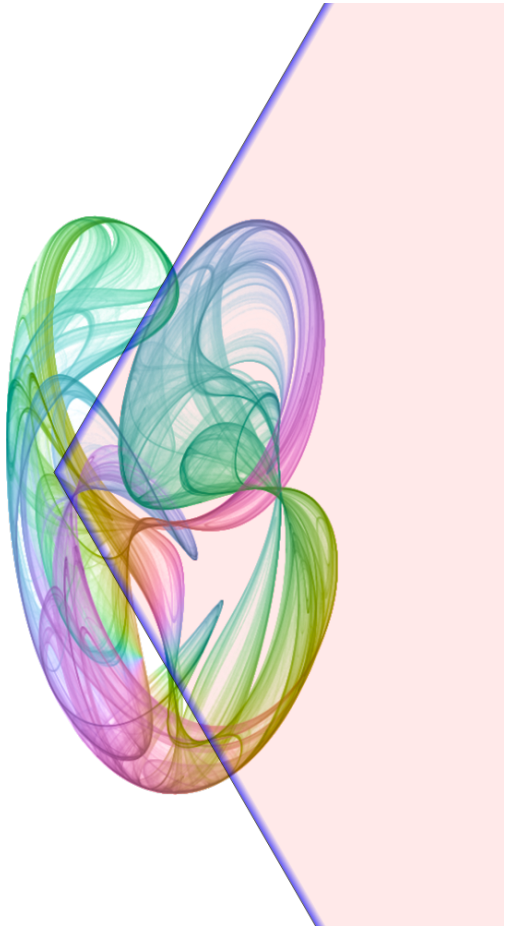
How to restrict iterations to the orbifold interior?

- Each iteration
- $z_{n+1} = F(z_n)$
- is composed with symmetrization function $z_{n+1} = S(F(z_n))$

Symmertization $S(z)$ moves any point on the plane into the corresponding point in the interior of the fundamental domain.

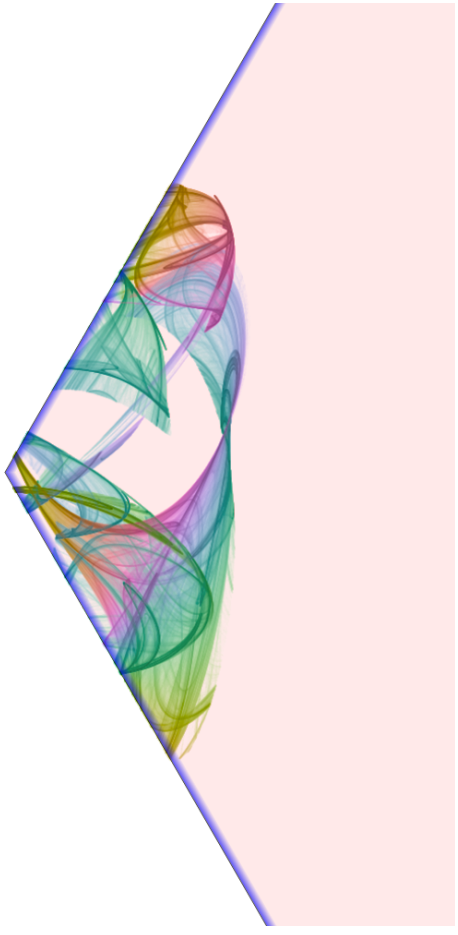
In this case it is clockwise rotation $\frac{2\pi}{3}$.

Dynamics on Orbifold 3



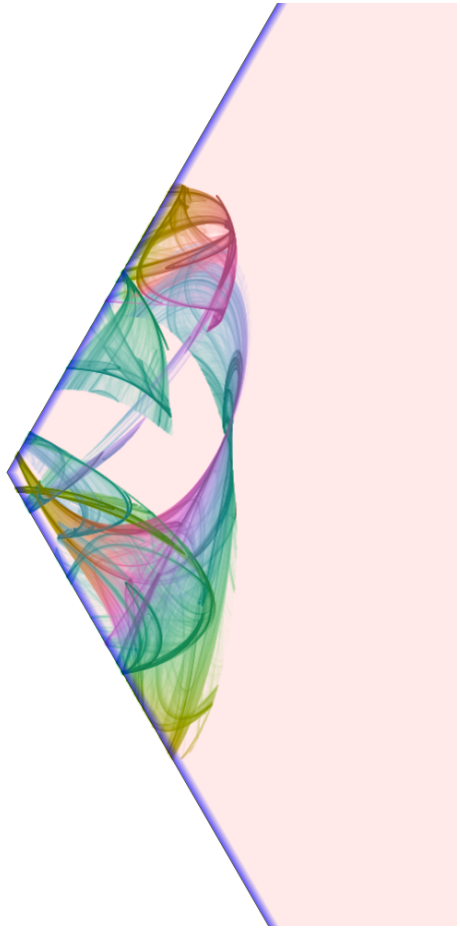
Original attractor without symmetrization

Dynamics on Orbifold 4



- Attractor with Symmetrization
- $z_{n+1} = S(F(z_n))$

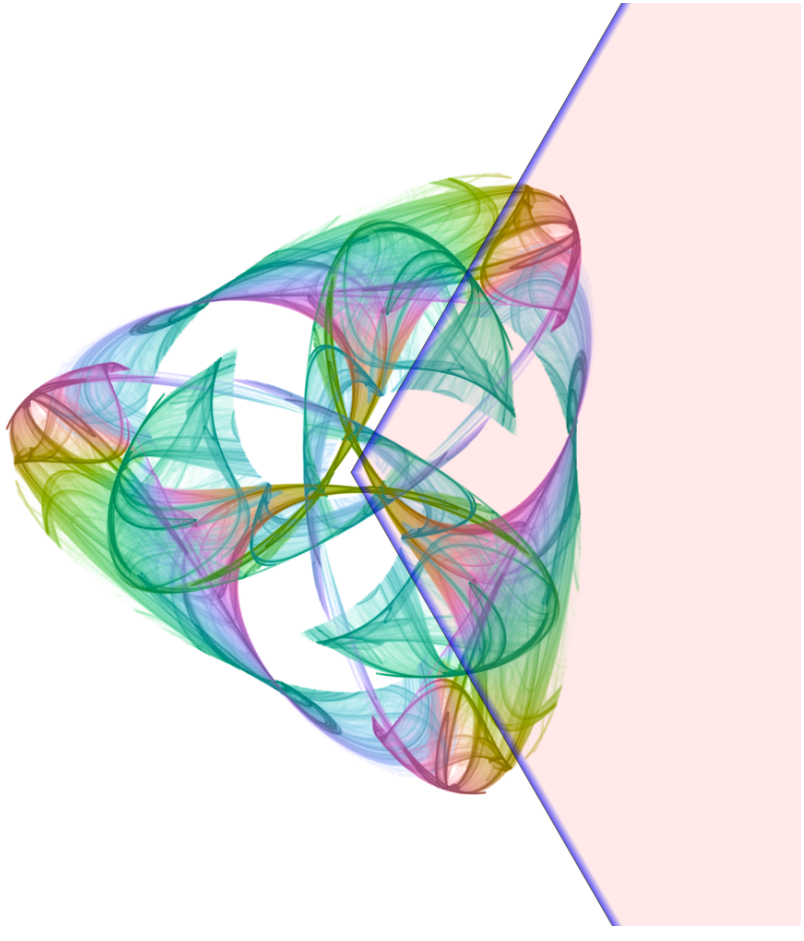
Dynamics on Orbifold 5



The new attractor in the fundamental domain is abruptly cut on the the boundary of the fundamental domain.

Is it?

Dynamics on Orbifold 7



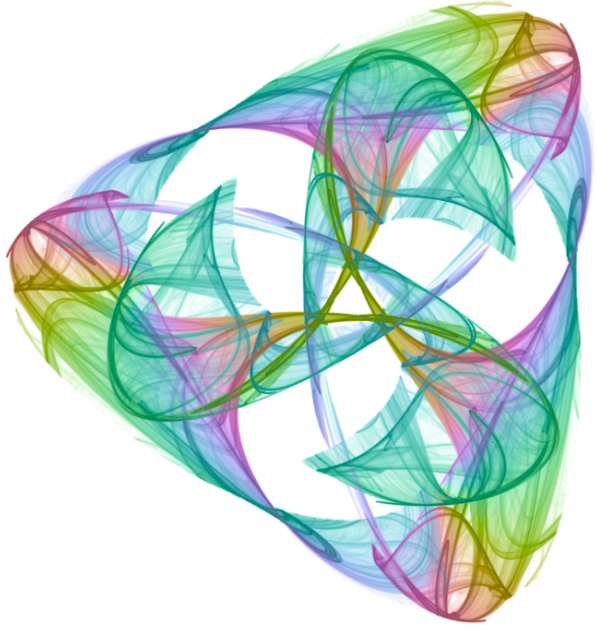
The new attractor in the fundamental domain is abruptly cut on the boundary of the fundamental domain.

Is it?

It's not!

After we tile the plane with the images of the fundamental domain the attractor is perfectly **continuous** and **smooth**!

Dynamics on Orbifold 8



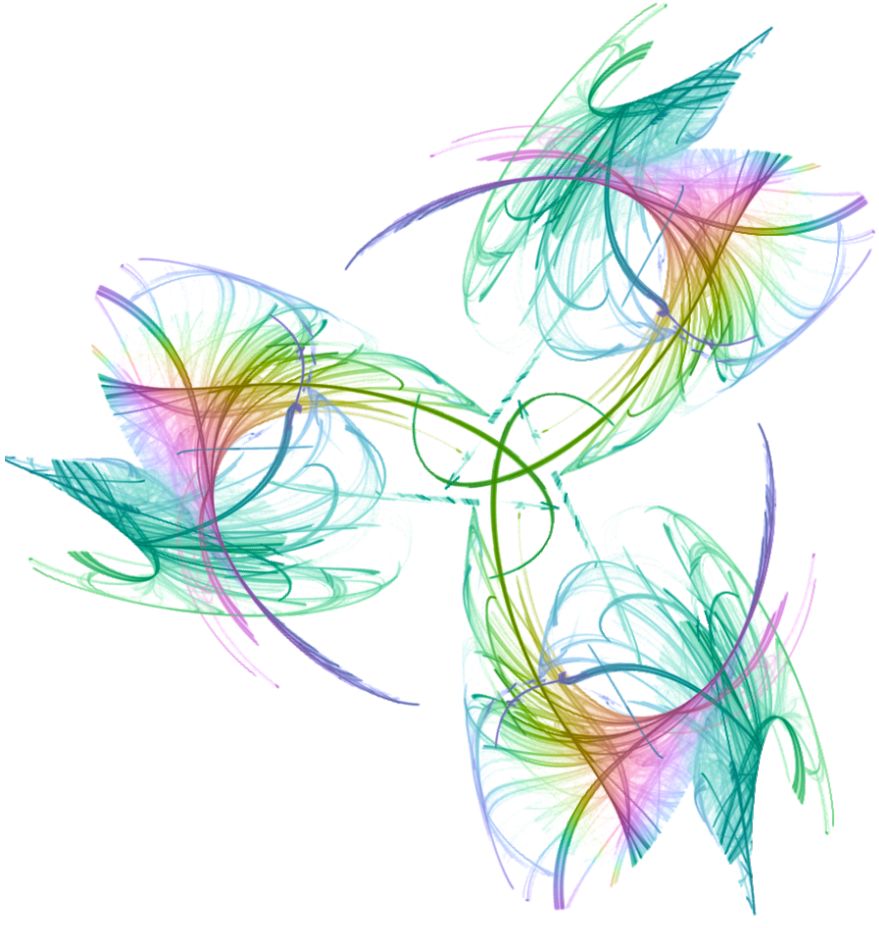
The attractor is perfectly **continuous** and **smooth**!

It is impossible to locate the former boundaries of the fundamental domain!

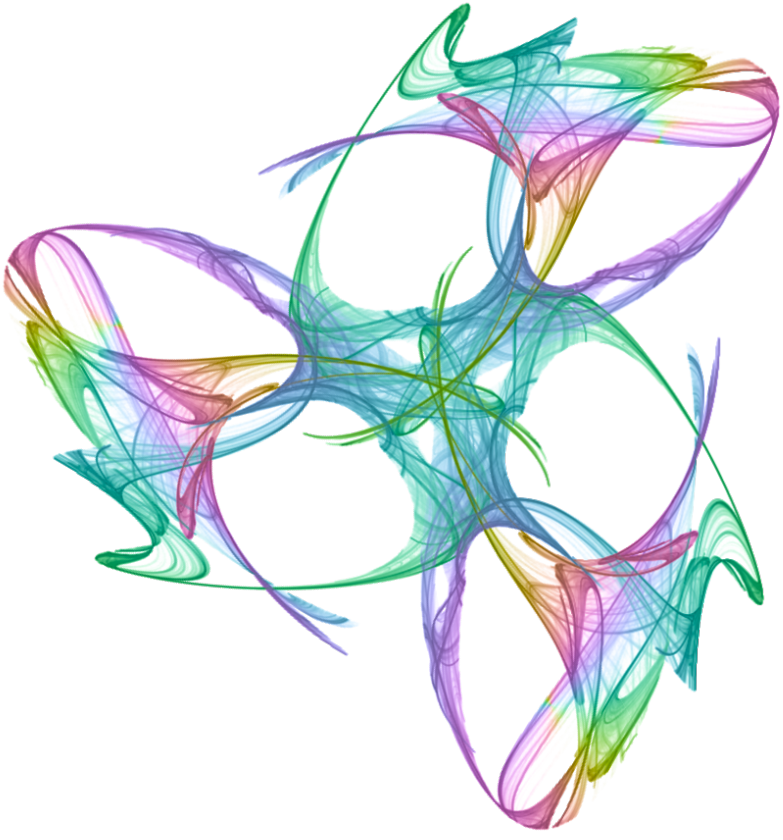
It is visual proof that the orbifold of tri-fold symmetry (the cone) has no boundary.

The places where we've cut the surface of the cone is completely arbitrary.

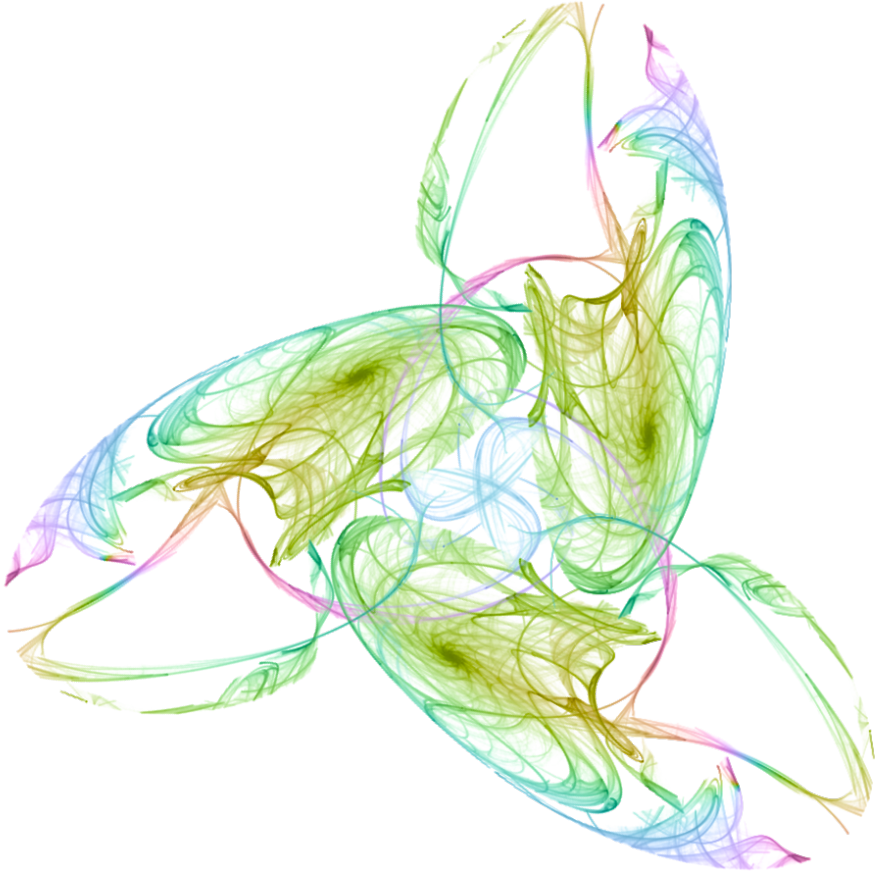
Example: Rosette symmetry 3



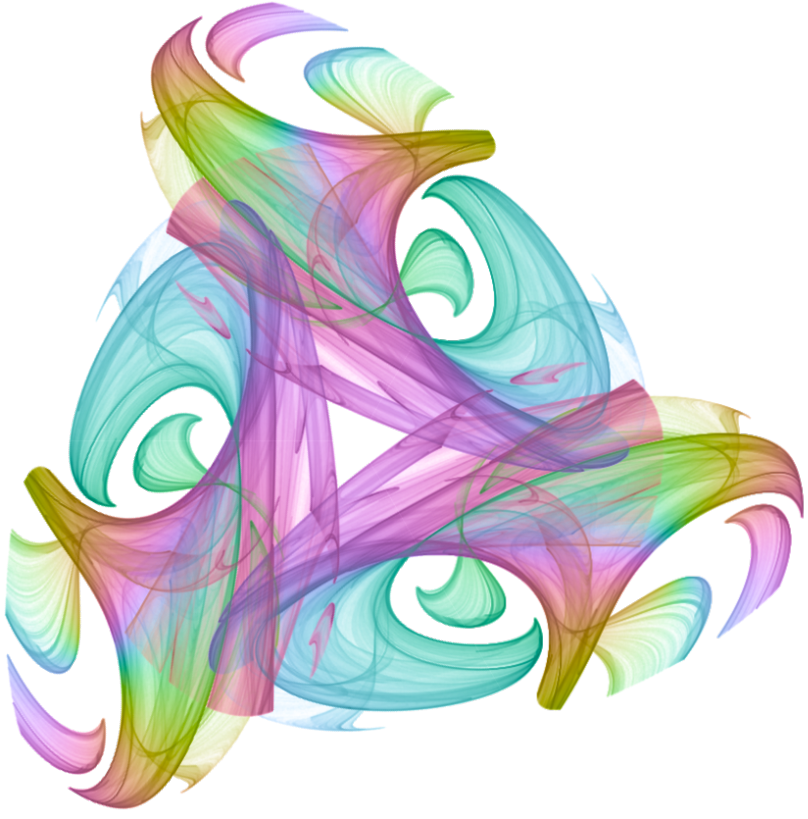
Example: Rosette symmetry 3



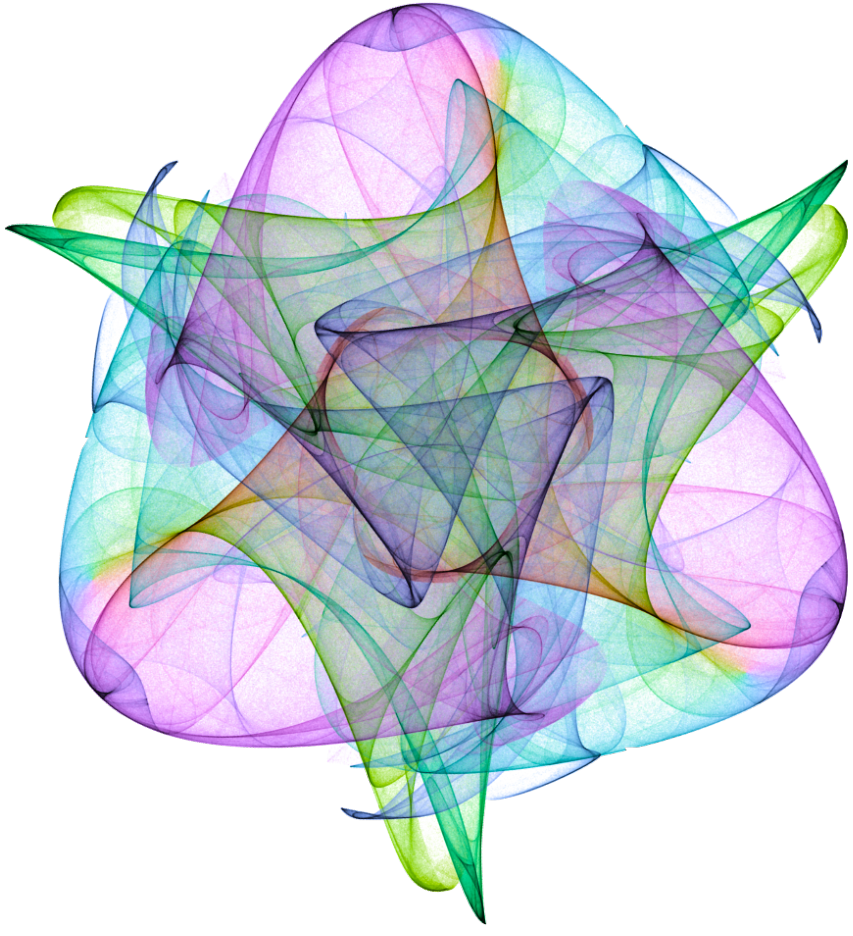
Example: Rosette symmetry 3



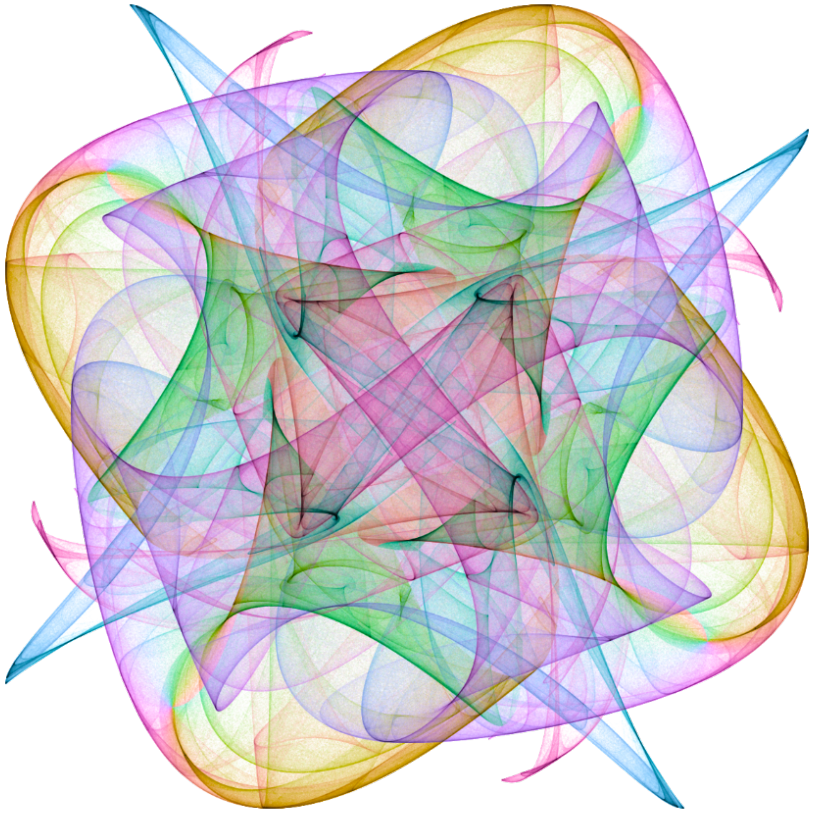
Example: Rosette symmetry 3



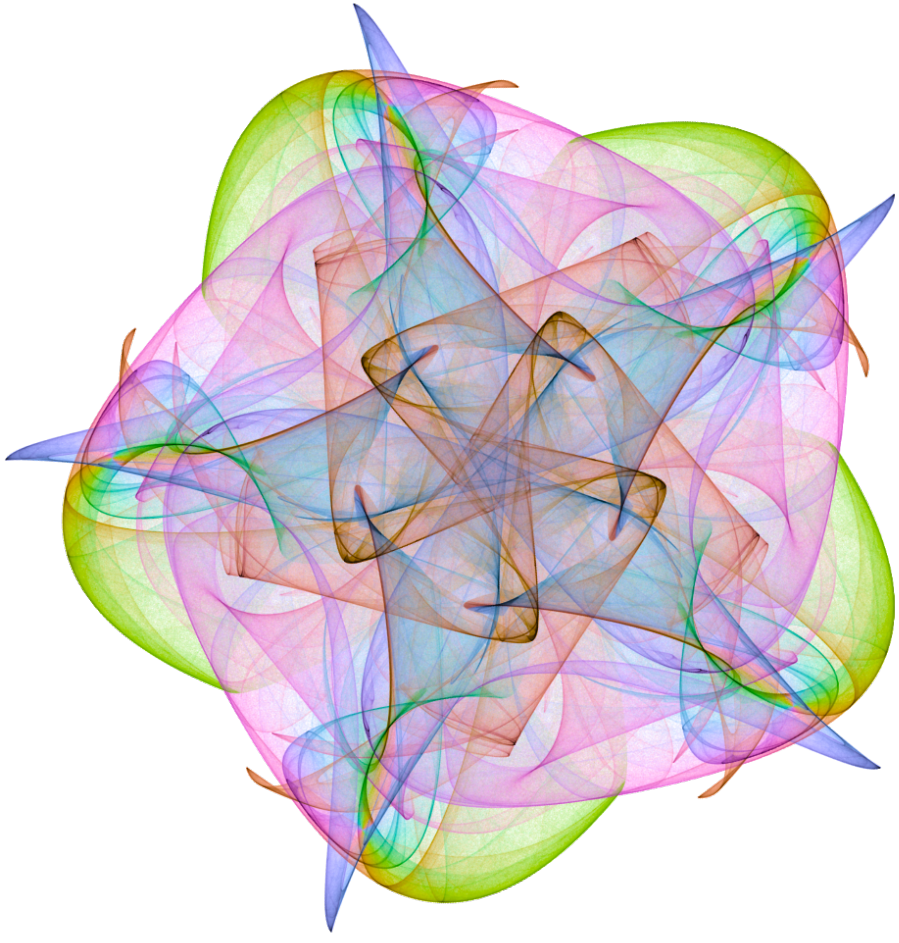
Example: Rosette symmetry 3



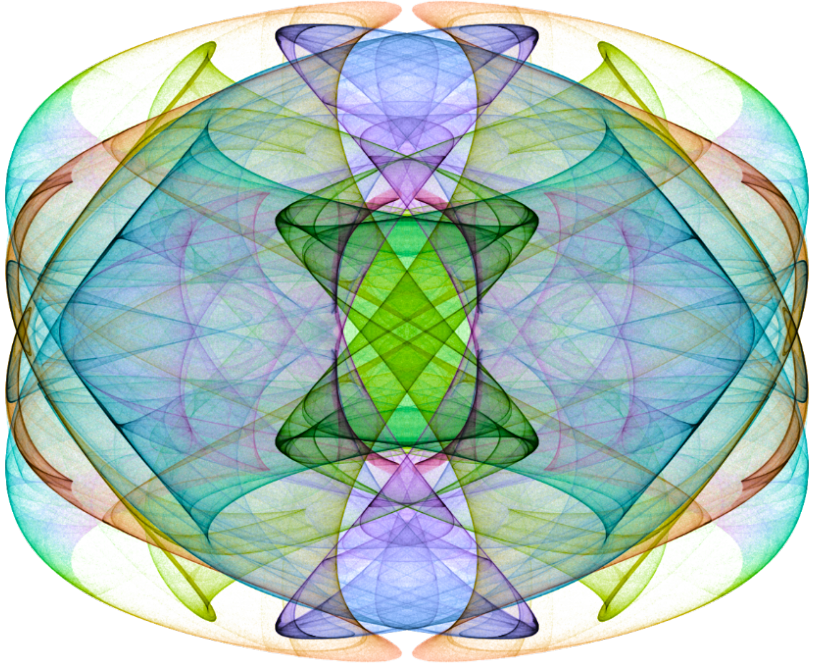
Example: Rosette symmetry 4



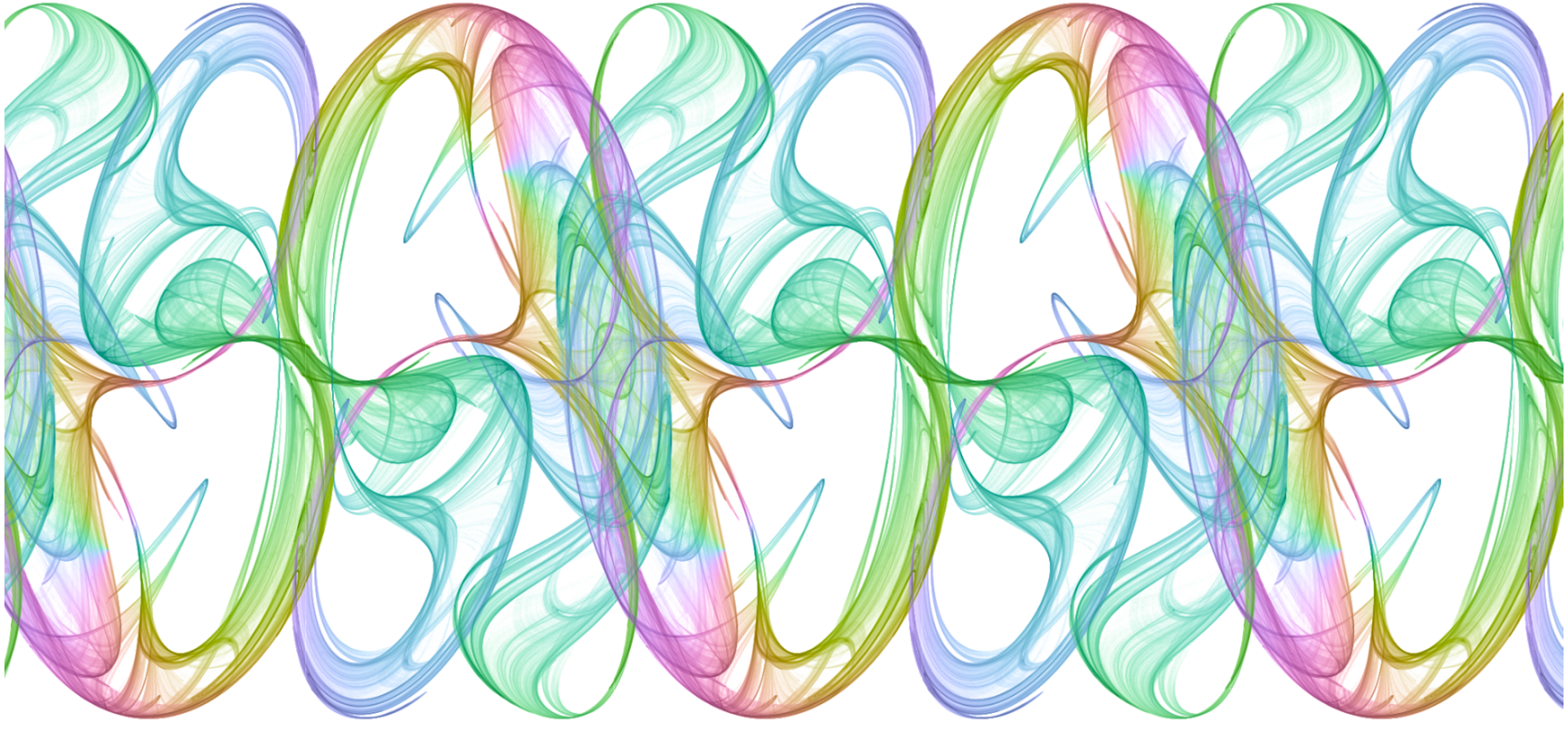
Example: Rosette symmetry 5



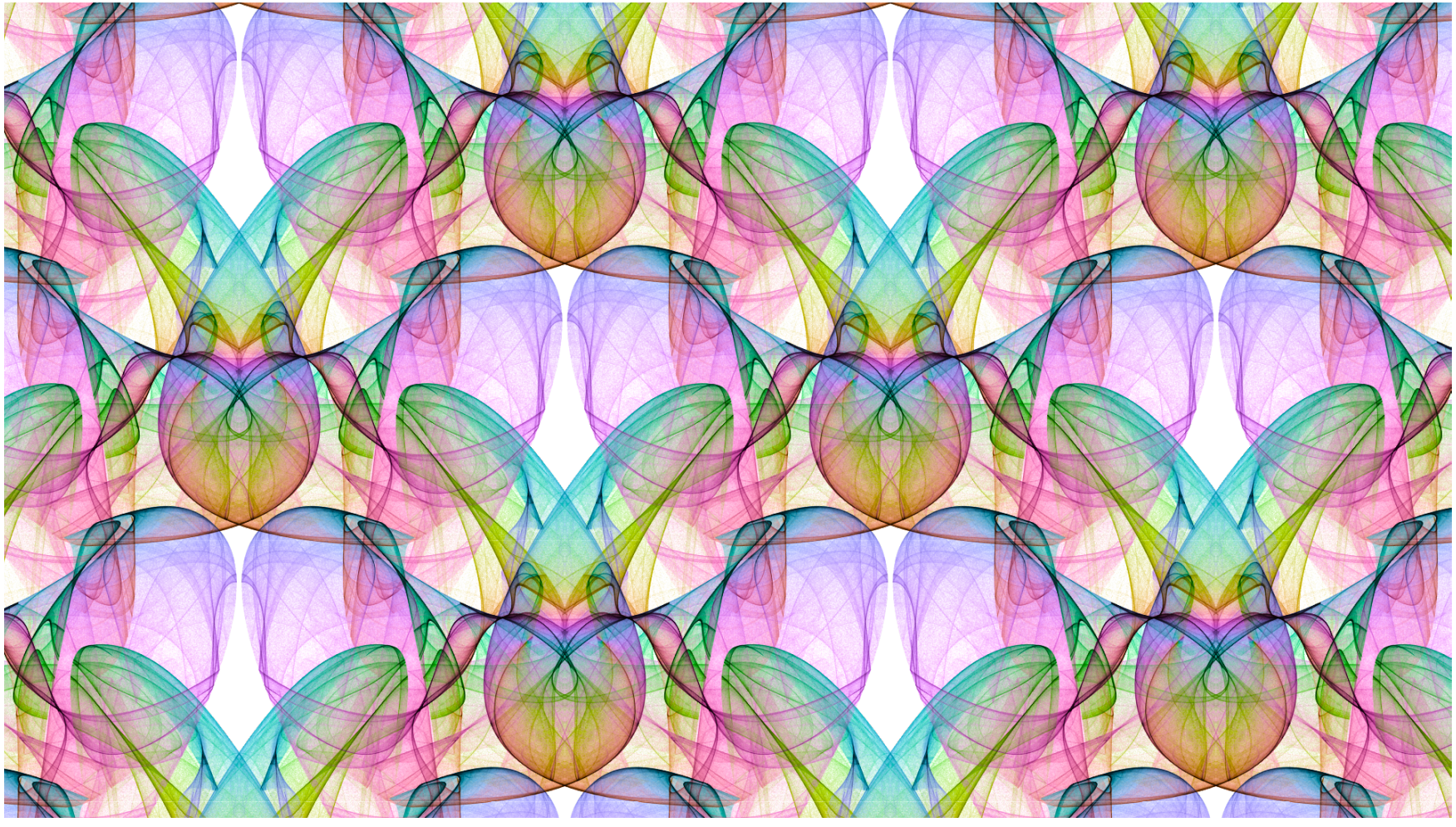
Example: Rosette symmetry *2



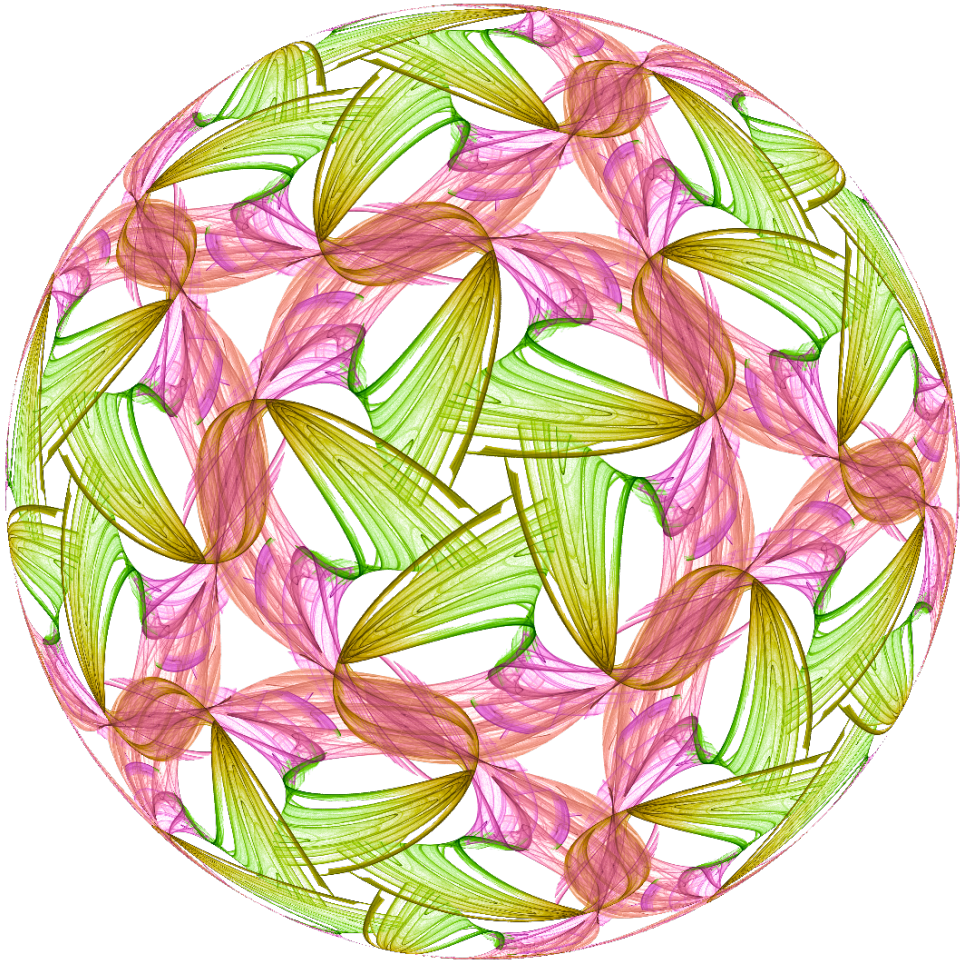
Example: Frieze symmetry 22∞



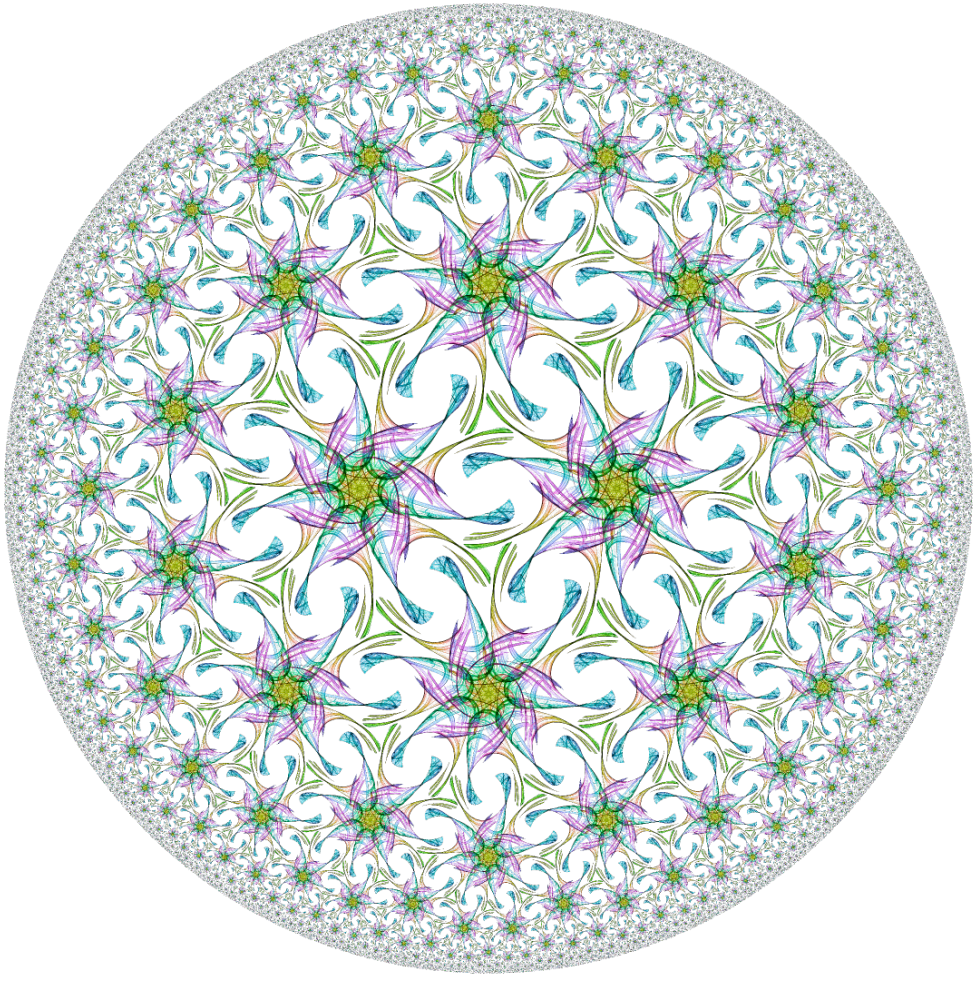
Example: Wallpaper symmetry *X



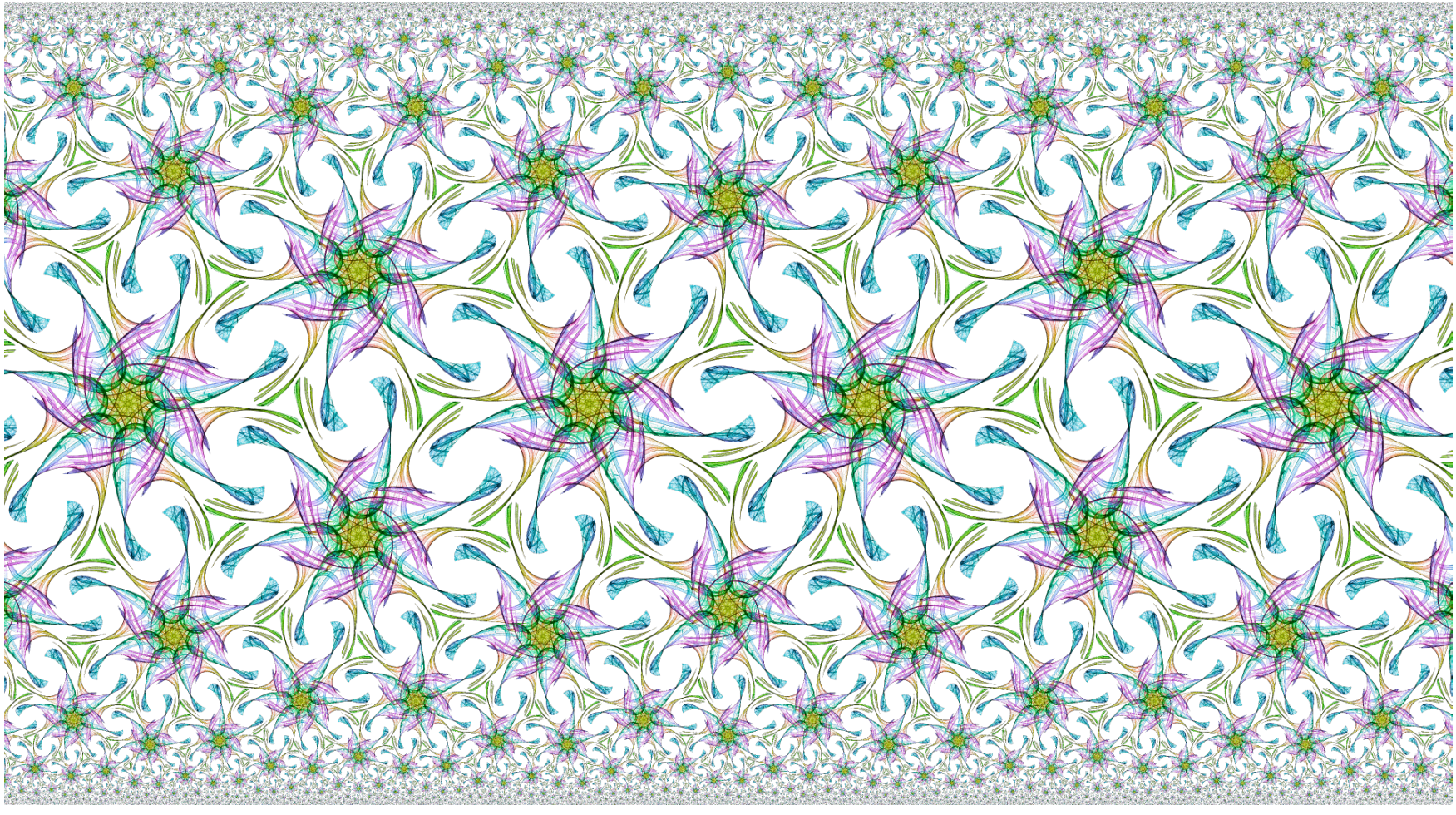
Example: Sphercal symmetry 235



Example: Hyperbolic symmetry 237, circle model



Example: Hyperbolic symmetry 237, band model



video 1



0:00 / 0:20

video 2



video 3



video 4



The Tool: SymmHub

- SymmHub - open source toolkit for experiments with symmetries
- https://symmhub.github.io/SymmHub/
- Creates interactive visualization using web browser
- Uses standard *HTML, Javascript and WebGL*
- Runs on desktops, tablets and phones
- **SymmHub created by:**
- *Vladimir Bulatov*
- *Chaim Goodman-Strauss*
- *Scott Vorthmann*



Looking for new victims of the symmetry hammer

